7th Grade Competition

Bergen County Academies
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1. A student has to compile 250 questions for a math competition. Since she is a procrastinator, she hasn't started yet. If she has 10 days left, how many questions must she write per day to complete the competition in time?
   Answer: 25
   If she has to write 250 questions in 10 days, she has to write $250/10 = 25$ questions per day.

2. What is the area of a triangle with base 7 and height 4?
   Answer: 14
   The formula for the area of a triangle is $a = \frac{1}{2}bh$, where b is the length of the base of the triangle and h is the height of the triangle. Thus the area of the triangle is $(\frac{1}{2})(7)(4) = 14$

3. Let $\upsilon(x)$ be defined as $5x + 20$ for every real number x. What is the value of $(\upsilon(3) + \upsilon(5))$?
   Answer: 80
   $(\upsilon(3) + \upsilon(5)) = 5 * 3 + 20 + 5 * 5 + 20 = 35 + 45 = 80$

4. Find $|4 - 7| + |7 - 4|$
   Answer: 6
   Notice that $|4 - 7| = |-3| = 3$ and $|7 - 4| = |3| = 3$ then $|4 - 7| + |7 - 4| = 3 + 3 = 6$

5. An extremely thirsty fish can drink one cubic yard of water in exactly 30 minutes. How many minutes would it take the fish to drink 36 cubic feet of water?
   Answer: 40
   36 cubic feet $= (36/(3 * 3 * 3) = 4/3)$ cubic yards, consumed in $(4/3)(30) = 40$ minutes.

6. $\frac{3+6+9+12+...+291+294}{4+8+12+16+...+388+392} = ?$
   Answer: 3/4
   Factor the numerator and denominator to get $\frac{3(1+2+3+...+97+98)}{4(1+2+3+...+97+98)} = 3/4$

7. Peter can mow a lawn that measures 600 square yards in 1.5 hours. At this rate, how many minutes would it take him to mow a lawn that measures 600 square feet?
   Answer: 10
   There are $3 * 3 = 9$ square feet in a square yard. Thus, Peter can mow 600 square feet in $1.5/9$ hours $= 10$ minutes

8. Suppose that it takes Chirag on average 45 minutes to paint a fence. It takes Trent on average 30 minutes to paint the same fence. If Chirag and Trent worked together to paint the fence, how many minutes would it take them to finish?
   Answer: 18
   Let the total time it takes Chirag and Trent to paint the fence be t. In one minute, Chirag can paint $1/45$ of the fence and Trent can paint $1/30$ of the fence. Then in t minutes, Chirag paints $t/45$ of the fence and Trent paints $t/30$ of the fence; together they paint $t/45 + t/30$ of the fence

in t minutes. Since we want them to finish, $t/45 + t/30 = 1$. Solving for t, we obtain $5t/90 = 1$, or $t = 18$ minutes.

9. Brian has 100 feet of fencing. He will use the fencing to enclose a rectangular play area for his puppy. What is the maximum number of square feet he can enclose?
   Answer: 625
   The maximum area is attained when we have a square of side x. $4x = 100$. Hence $x = 100/4 = 25$ and the area is $25^2 = 625$.

10. Find the sum of the numbers from 1 to 15 inclusive.
    Answer: 120
    $1 + 2 + 3 + ... + 14 + 15 = (1 + 15) + (2 + 14) + (3 + 13) + ...(7 + 9) + 8 = (7 * 16) + 8 = 120$

11. Riding their bicycles, Alex and Brian leave from two different places at the same time and ride directly toward each other. Alex rides at 10 mi/h and Brian rides at 8 mi/h. If they meet after 40 minutes of riding, how far away were they at the beginning?
    Answer: 12
    Each hour, Alex and Brian get $10 + 8 = 18$ miles closer to each other. If it only takes 2/3 of an hour for them to meet, then their original distance was (2/3)(18 miles) = 12 miles.

12. It takes 20 of Beowulf's men to beat a dragon in battle. If Beowulf has 170 men, how many dragons could they beat in battle?
    Answer: 8
    Since it takes 20 of Beowulf's men to defeat one dragon, Beowulf's men can handle $170/20 = 8.5$ dragons in a battle. However since half-dragons do not count, Beowulf's men can handle 8 dragons in battle.

13. If 55% of a number is 935, what is 70% of that number?
    Amswer: 1190
    Let x be the number. Then $x = \frac{100}{55} * 935 = 1700$. Hence the answer is: $\frac{70}{100} * 1700 = 1190$.

14. There are 9 teams in a school district competition. Each team plays each other team once. What is the total number of games played in the competition?
    Answer: 36
    Each of the 9 teams plays 8 other teams. This gives $9 * 8 = 72$ games. However, this counts every game twice, hence the answer is: $72/2 = 36$ games.

15. Blocks of molding clay are 9 inches by 6 inches by 3 inches. How many whole blocks are needed to mold a cylindrical sculpture 13 inches high and 6 inches in diameter? Use 3.14 as an approximation of Pi.
    Answer: 3
    v_1 =volume of a block $= 3*6*9 = 162$. v_2 =volume of a sculpture $= 3.14*(3*3)*13 = 367.38$. $v_2/v_1 = 2.26778$. Hence the answer is 3.

16. A club found that it could acheive a ratio of 2 adult members for every minor member either by inducting 24 adults or by expelling x minors. Find x.
    Answer: 12
    Let the number of adults and the number of minors be represented by A and M, respectively. Then $\frac{A+24}{M} = \frac{2}{1}$, so $2M - 24 = A$. Also, $\frac{A}{M-x} = \frac{2M-24}{M-x} = \frac{2}{1}$, leading to $x = 12$.

17. After reading Shakespeare, Rachel decides to speak in iambic pentameter. This means that she speaks in lines with 5 sets of 2 syllables each. Ben rudely interrupts her after she has only

spoken 13 lines and 70% of her 14th line. How many syllables has she said?
Answer: 137
She said $13 + 7/10$ lines. Each line has $2 * 5 = 10$ syllables so she said $13.7 * 10 = 137$ syllables.

18. Two passenger trains traveling in opposite directions meet and pass each other. Each train is 1/24 miles long and is traveling at 50 miles per hour. How many seconds after the front parts of the trains meet will their rear parts pass each other?
Answer: 3 or 3 seconds
The distance traveled is just the length of the train, which is 1/24 miles. Since the train is traveling at 50 miles per hour, (50 miles per hour)(time) = 1/24 miles, which means that the time is 1/1200 hours = 3 seconds .

19. What is the 100th number in the arithmetic sequence 1, 4, 7, 10, 13, ...?
Answer: 298
The first term is $3 \times 1 - 2$, the second is $3 \times 2 - 2$, and in general the nth term is $3n - 2$. The 100th term is $3 \times 100 - 2 = 298$.

20. The sum of 4 consecutive odd numbers is 216. Find the smallest of these numbers.
Answer: 51
Let the four consecutive odd numbers be $x, x+2, x+4$, and $x+6$. Then $4x+12 = 216 \rightarrow x = 51$.

21. Suppose that Michelle wishes to buy a laptop. The HP laptop costs \$1500 with an immediate 10% discount. Michelle can also later cash a \$100 rebate for the HP laptop. On the other hand, the Toshiba laptop costs \$1600 with a 25% discount, but no rebate. Assume that Michelle lives in a sales-tax-free nation. What is the price of the cheaper laptop?
Answer: 1200
HP: With the 10% discount, the laptop costs $\$1500 \times 90\% = \$1350$. Subtracting the \$100 rebate, the cost comes to $\$1350 - \$100 = \$1250$. Toshiba: With the 25% discount, the laptop costs $\$1600 \times 75\% = \$1200$. Hence, the Toshiba is cheaper, and its price is \$1200

22. What is the least number of people you could have in a group and still be guaranteed that at least 7 of them have birthdays in the same month?
Answer: 73
To force n people have birthdays in the same month, we need at least $12(n - 1) + 1$ people (by pigeonhole principle, if no n people are born in the same month, there can be at most $n - 1$ people for each month). Hence the answer is $12 * (7 - 1) + 1 = 73$.

23. Given that $5x + 7y = 9$ and $7x + 5y = 63$, what is the value of $x + y$?
Answer: 6
Add the two equations to get $12 * (x + y) = 9 + 63 = 72 = 12 * 6$ Hence $x + y = 6$.

24. The girls in physical education class sat around a large circle and spaced themselves evenly. To form teams, the instructor asked them to count off $1, 2, 3, 4, \ldots$ When they were finished counting, the girl who was 21st was sitting directly across from the girl who was 7th. How many girls were sitting around the circle?
Answer: 28
Since the two girls were sitting across from each other, we can presume that the people between them represent half of the total number of people. The difference of 21 and 7 is 14, which doubled equals 28.

25. If $7 \leq x \leq 12$, what is the value of $||x - 3| + |x - 24||$?
Answer: 21
Since $x \leq 24, |x - 24| = 24 - x$. $||x - 3| + |x - 24|| = |x - 3 + 24 - x| = 24 - 3 = 21$.

26. It takes Carrie 40 minutes to walk between her home and her school. One morning she walked half way to school and remembered that she had left her calculator at home. She ran home. It took 5 minutes to find her calculator when she got home. Then she ran all the way to school. She runs twice as fast as she walks. How many minutes more than usual did it take for her to get to school?
Answer: 15
$40 \times \frac{1}{2} + 20 \times \frac{1}{2} + 5 + 20 - 40 = 15$.

27. At age 8, Christine decided to start saving money from her allowance. She saved \$2 a month the first year, \$3 a month the second year, \$4 a month the third year, etc. She is turning 18 today. How much money has Christine saved so far?
Answer: 780
She has saved $(2*12)+(3*12)+(4*12)+...+(10*12)+(11*12) = 12(2+3+...+9+10+11) = 12*65 = \$780$.

28. On hypotenuse $\overline{AB}$ of right triangle $\triangle ABC, D$ is the point for which $\overline{CB} = \overline{BD}$. If $m\angle B = 40$ degrees, find $m\angle ACD$.
Answer: 20
In isoceles triangle $\triangle BCD$, $m\angle BCD = m\angle BDC = 70$, so $m\angle ACD = 90 - 70 = 20$.

29. Marina selects 2 numbers at random from 1 to 8. If she can choose the same number twice, what is the probability that the sum of two numbers selected is 5?
Answer: $\frac{1}{16}$
Here are all the pairs that add up to 5; $(4,1),(2,3),(3,2),(1,4)$. There are 4 of those and there are 64 total possibilities. So the answer is: $4/64 = 1/16$.

30. The lengths of the sides of a triangle are $3, 4$, and 6. What is the least possible perimeter of a similar triangle, one of whose sides has a length of 12?
Answer: 26
The least possible perimeter occus when 12 is the longest side. The other sides are thus 6 and 8. The perimeter would then be $6 + 8 + 12 = 26$

31. A point P is chosen inside the square $ABCD$. What is the probability that the angle APB is obtuse? Express your answer as a decimal to the nearest hundredths. Use $\pi = 3.14$
Answer: 0.39
The point P must be inside the semi-circle of diameter AB inside the square. Let us suppose that R is the radius of this semi-circle. The area of this semi-circle is $\pi R^2/2$. The area of the square is $(2R)^2 = 4R^2$. Hence the answer is: $(\pi R^2/2)/(4R^2) = \pi/8 \approx 0.39$.

32. Suppose the prime factorization of 6! is $2^a \times 3^b \times 5^c$. What is $a + b + c$?
Answer: 7
$6! = 6*5*4*3*2*1 = (2 \times 3)(5)(2^2)(3)(2) = 2^4 \times 3^2 \times 5$ so $a + b + c = 4 + 2 + 1 = 7$.

33. Each person in Lauren's math class shakes hands with each of the others exactly once, and 120 handshakes are exchanged altogether. How many people are in her class?
Answer: 16
Call the number of people x. Then there are $\binom{x}{2} = x(x-1)/2 = 120 \rightarrow x = 16$ people in Lauren's math class.

34. Given that $81^m = 3$ and $m^n = 16$, evaluate mn.
Answer: -1/2
Notice that $3^4 = 81$, so $m = 1/4$. Now $(1/4)^n = 16$ implies $n = -2$. Hence $mn = -1/2$

35. Define a 3-digit number n to be a *multiplenumber* if the hundreds digit is the product of the tens and ones digit. The number 632 is a *multiplenumber* since $6 = 3*2$. How many three-digit multiple numbers less than 500 exist?
Answer: 8
The multiple numbers less than 500 are $111, 212, 221, 313, 331, 414, 422, 441$.

36. Yoonjoo wants to sell all of her 60 pencils in combinations of 5 or 3 or both. In how many ways can the pencils be grouped?
Answer: 5
Here are all the possibilities: ((0*5,20*3), (3*5,15*3), (6*5,10*3), (9*5,5*3), (12*5,0*3)).

37. Ian has a container holding 60 quarts of mixtures of 30% NaCl and 70% H_2O. Ian has a second container holding 40 quarts of mixtures of 50% NaCl and 50% H_2O. If he mixes them, what percent, to the nearest whole percent, the mixture will be NaCl?
Answer: 38
$100 * \frac{60*(\frac{30}{100})+40*(\frac{50}{100})}{60+40} = 38$ to the nearest whole percent.

38. On planet Holbrook, there are as many days in a week as there are weeks in a month. The number of months in a Holbrook year is twice the number of days in a month. If there are 1250 days in a Holbrook year, how many months are there?
Answer: 50
Let x be the number of days in a week, which equals the number of weeks in a month. Then, x^2 is the number of days in a month, and $2x^2$ is the number of months in a year. For the number of days in a year, multiply the number of days in a month by the number of months in a year: $x^2 * 2x^2 = 1250$, which means that $x^2 = 25$ and $2x^2 = 50$

39. The obtuse angle of an isosceles triangle is bisected and each resulting angle is 59 degrees larger than a base angle. How many degrees are in the measure of the obtuse angle?
Answer: 149
Let x be the measure of the obtuse angle. The measure of the base angle will then be $\frac{x}{2} - 59$. Then solve $x+(x/2-59)2 = 180 \rightarrow x+x-2*59 = 180 \rightarrow 2x = 180+2*59 \rightarrow x = 90+59 = 149$.

40. Let ABCD be a square of side length 12 units. Let E, F, G and H be the midpoints of AB, BC, CD and DA respectively. What is the number of square units in the area of the triangle BHG?
Answer: 54
$Area(BGH) = 144 - Area(ABH) - Area(BCG) - Area(DHG)$. Each of the parts are right triangles with areas $144/4, 144/4, 144/8$ respectively, so the total is $144 - (36 + 36 + 18) = 54$.

41. An integer is represented by a two-digit base 10 numeral. If three times the sum of its digits is added to the integer, the result is the original integer with its digits reversed. For example, $12 + 3 * (1 + 2) = 21$. Including 12, how many such positive integers exist?
Answer: 4
If the integer is $10t + u$ [where t, u are digits], then $3(t + u) + 10t + u = 10u + t$, which simplifies to $u = 2t$. The solutions to this are $12, 24, 36$, and 48.

42. Find all real values x that satisfy

$$\frac{x^3 - x^2 - x + 1}{x^3 - x^2 + x - 1} = 0$$

Answer: -1
For the fraction to be 0, the numerator must be 0 but the denominator cannot. The top is $(x - 1)(x^2 - 1)$ and the bottom is $(x^2 + 1)(x - 1)$. Hence, the quotient is $(x^2 - 1)/(x^2 + 1)$ which is zero at $x = \pm 1$. However, at $x = 1$ the denominator is 0, so the answer is $x = -1$.

43. Let $f(x) = x^2, g(x) = 2x, \phi(x) = f(g(x)) - g(f(x))$. What is $\phi(10)$?
Answer: 200
$f(g(x)) = (2x)^2 = 4x^2$ and $g(f(x)) = 2x^2$ so $\phi(x) = 4x^2 - 2x^2 = 2x^2$ and $\phi(10) = 200$.

44. Two standard dice are rolled. What is the probability that the product of the numbers on the top faces will be greater than 6? Express your answer as a common fraction in lowest terms.
Answer: 11/18
The possibilities are: $(2, [4 \ldots 6]); (3, [3 \ldots 6]); (4, [2 \ldots 6]); (5, [2 \ldots 6]); (6, [2 \ldots 6])$. There are $3 + 4 + 5 + 5 + 5 = 22$ pairs out of $6 \times 6 = 36$ pairs, for a probability of $22/36 = 11/18$.

45. What is the smallest integer that $4,320,000$ must be multiplied by to get a number with exactly eight terminating zeroes?
Answer: 625
The prime factorization of 4320000 is $2^8 * 3^3 * 5^4$. We need eight 2-5 pairs, which means we need 4 fives, or $5^4 = 625$.

46. What is the probability that two consecutive positive integers $x < y \leq 100$ satisfy $x \times y \leq 100$?
Answer: 1/11
$y = x + 1$, so $1 \leq x \leq 99$. $9 * 10 < 100 < 10 * 11$ so $x \leq 9$. Thus, there are 9 good choices for x, so the probability is $9/99 = 1/11$.

47. A chord of the larger circle of two concentrics circles is tangent to the smaller (inner) circle and measures 14 inches. The number of square inches in the area outside the smaller circle and inside the larger circle can be expressed as $x\pi$. Find x.
Answer: 49
Let R be the radius of the largest circle and r be the radius of the smallest circle. Then Let N be the point the chord is tangent to the smaller circle and M and P be the points where this chord intersects the larger circle. Let O be the common center of the two circles. Then in the right triangle ONP, one has $NP^2 = R^2 - r^2 = (14/2)^2 = 49$. The area between the two circles is: $\pi(R^2 - r^2) = 49\pi$. Hence the answer is 49.

48. How many ways can the letters of the word $NUMBER$ be scrambled so that the first and the last letters are both vowels?
Answer: 48
Notice that $NUMBER$ has 2 distinct vowels, U and E, which must be at the beginning and end. Hence the answer is $2! * 4! = 48$.

49. Ten tennis players participate in a tournament in which each player plays every other player exactly once. After each match, the two players shake hands. Then, both players shake hands with the umpire. After all of the matches, how many handshakes have been exchanged?
Answer: 135
Each match, three handshakes take place: the one between the players, the one between the winner and the umpire, and the one between the loser and the umpire. Since each match occurs between two players, there are $\binom{10}{2} = \frac{10!}{2!8!} = \frac{10*9}{2} = 45$ matches. Thus, there are $45 * 3 = 135$ handshakes after all of the matches.

50. If $a > 0$ and if $(x+1)(x+2)(x+3)(x+4)+1 = (ax^2+bx+c)^2$, find the ordered triple (a, b, c).
Answer: (1,5,5)
Rearranging, $(x+1)(x+2)(x+3)(x+4)+1 = [(x+2)(x+3)][(x+1)(x+4)]+1 = (x^2+5x+6)(x^2+5x+4)+1 = (y-1)(y+1)+1 = y^2$, where $y = x^2+5x+5$.