8th Grade Competition

Bergen County Academies Math Competition
21 October 2007

1. A student is compiling 250 questions for a math competition. She asked each student to write at least 5 questions with solutions. She accepted a question if and only if it came with a solution and was unique. Out of 125 math team members, only 15 wrote questions. One student wrote 42 questions with solutions. One student wrote 12 questions with solutions. Seven students gave 5 questions each without solutions. Six students gave 2 questions each with solutions. However, she received 3 repeats. How many questions did she have to produce herself?
   Answer: 187
   Two students wrote $42 + 12 = 54$ usable questions. Seven students gave 5 questions without solutions, which renders them unusable. Six students wrote $(6*2) = 12$ usable questions, which means that she received 66 total questions. However, since 3 of them were repeats, she received $66 - 3 = 63$ usable questions. Thus she has to write $250 - 63 = 187$ questions.

2. $-51 - (-52) =?$
   Answer: 1
   $-51 - (-52) = -51 + 52 = 1$

3. Joyce is making a classroom model of the solar system. Mercury, the smallest planet, has a diameter of 3,000 miles. Jupiter, the largest planet, has a diameter of 88,700 miles. She decided to use a pea measuring 0.5 cm in diameter as the Mercury model. Using the same scale, what size, to the nearest whole centimeter, should the diameter of the Jupiter model be?
   Answer: 15
   $\frac{d}{88700} = \frac{0.5}{3000} \rightarrow 3000 \times d = 0.5 \times 88700 \Rightarrow d = \frac{0.5\times 88700}{3000} = 14.78$ cm, which rounds to 15 cm

4. $\frac{2}{5} = \frac{8}{?}$
   Answer: 20
   $\frac{2}{5} * \frac{4}{4} = \frac{8}{20}$

5. Pavel wants to buy a CD player that costs \$56 including tax. He gets \$10 a week for his allowance. He spends \$3.50 a week and saves the rest. How many weeks will it take him to save enough money to buy the CD player?
   Answer: 9
   Pavel saves $10 - 3.50 = 6.50$ each week. Thus, it will take $56 \div 6.50 = 8.62$ weeks. Since it will take more money than Pavel gets in 8 weeks, it will take 9 weeks to save enough money.

6. If $v + w = w$, what is the value of v?
   Answer: 0
   Subtract w from both sides: $v + w - w = w - w \rightarrow v = 0$.

7. Vincent runs at a constant rate of 7 meters per second. At the beginning of a race, he is 350 meters from the finish line. How many seconds will it take Vincent to reach the finish line?
   Answer: 50
   Divide the distance that remains by the rate at which Vincent runs: $350 \div 7 = 50$ seconds.

8. A telephone call to India costs \$0.29 per minute. How long was a telephone call that cost \$1.45?
Answer: 5 minutes
$0.29x = 1.45 \rightarrow x = 1.45/0.29 = 5$

9. In how many ways can 4 socks be chosen from a drawer containing 9 socks of different colors?
Answer: 126
The socks are distinct, so we can choose 4 in $\binom{9}{4} = 9 * 8 * 7 * 6/(4 * 3 * 2 * 1) = 126$ ways.

10. What is the number halfway between $\frac{1}{11}$ and $\frac{1}{7}$?
Answer: $\frac{9}{77}$
The number halfway between is the average: $\frac{1}{2}(\frac{1}{11} + \frac{1}{7}) = \frac{9}{77}$.

11. A certain book has 500 pages numbered $1, 2, 3$, and so on. How many times does the digit 1 appear in the page numbers?
Answer: 200
The digit 1 appears in the ones place 50 times $(1, 11, 21, \ldots, 491)$. It also appears in the tens place 50 times $(10 - 19, 110 - 119, \ldots, 410 - 419)$. It occurs 100 times in the hundreds place $(100 - 199)$. So, the total number of 1's is $50 + 50 + 100 = 200$.

12. Marina spent two thirds of her money. Then she lost two thirds of the money that was left. Four dollars remained. How much money did Marina have in the beginning?
Answer: 36
After spending two thirds of her money, Marina had one third of it left. After losing two thirds of the one third she had left, she had $(1/3)(1/3) = 1/9$ of her original money left. If $1/9$ of her original amount of money was four dollars, then her original amount of money was $9 * 4 = \$36$.

13. What is the least common multiple of the numbers $6, 9$, and 25?
Answer: 450
$6 = 2 * 3, 9 = 3^2$, and $25 = 5^2$. Thus, their least common multiple is $2 * 3^2 * 5^2 = 450$.

14. When Ben has a "sharpie battle," he has a 1/3 chance of poking his opponent's arm and a 1/5 chance of poking their neck (neither affects the other). When he faces Yoonjoo, what is the probability that, per shot, he pokes her arm, her neck, and her arm again in that order?
Answer: 1/45
They are independent events, so $P(A \text{ and } B) = P(A) * P(B)$ applies. $P = 1/3 * 1/5 * 1/3 = 1/45$.

15. Robert has two watches, one which loses 6 seconds every 24 hours and one which gains 1 second per hour. He sets both of them to the correct time at 6 : 00 p.m. How many hours will pass before the positive difference between the time shown on both watches is 4 hours?
Answer: 11,520
After 24 hours, the difference will be 30 seconds. After $120 * 24$ hours, the difference will be 3600 seconds or one hour. Thus after $4 * 120 * 24$ hours, the difference will be 4 hours. So the answer is $4 * 120 * 24 = 11,520$

16. Ashley is 18 years old and Scott is twice her age. If Ashley's mom is 10 years older than Scott, how old is Ashley's mom?
Answer: 46
If Ashley is 18 years old and Scott is twice her age, then Scott is $18 * 2 = 36$ years old. Thus if Ashley's mom is 10 years older than Scott, she is $36 + 10 = 46$ years old.

17. $36^{1/2} = ?$
Answer: 6
$36^{1/2} = \sqrt{36} = 6$

18. Matt has a magic number basket. The only numbers that can be placed in the basket are numbers that have two, three, or four digits, all the digits must be odd, and the digits must increase from left to right. How many numbers can Matt place in the basket?
Answer: 25
For each subset of $(1, 3, 5, 7, 9)$ there is exactly one valid order. For example, the subset {1,5,3} admits 135 as a number. There are $\binom{5}{2} + \binom{5}{3} + \binom{5}{4} = 25$ subsets with 2,3, or 4 digits.

19. The arithmetic mean of 10 numbers is what percent of the sum of the same 10 numbers?
Answer: 10%
Let x be the arithmetic mean of the 10 numbers and y be the sum of the 10 numbers. Then $x = S/10$. The answer is $100 * (S/10)/S = 100/10 = 10$.

20. Twenty chess players hold a tournament in which each player plays one game with each of the other players. How many games are played altogether?
Answer: 190
The number of games is the number of pairs of players, $\binom{20}{2} = \frac{20!}{18!2!} = \frac{20*19}{2} = 19 * 10 = 190$.

21. In an isosceles triangle, the perpendicular bisector of one leg passes through the midpoint of the base. If the length of this leg is 10, how long is the base in simplest terms?
Answer: $10\sqrt{2}$
Since the perpendicular bisector of one leg passes through the midpoint of the base (which lies on the perpendicular bisector of the base), all three perpendicular bisectors intersect at that point. Hence, the intersection point is also the circumcenter of the triangle. Therefore, the length of the base is twice the radius of the circle (= diameter of the circle), the outer arc is 180 degrees, and the vertex angle is 90 degrees. The triangle is, in fact, an isosceles right triangle (side ratio $1 - 1 - \sqrt{2}$). If the leg length is 10 the base must be $10\sqrt{2}$.

22. Find the value of n if: $\log_2 \log_3 \log_4 2^n = 2$
Answer: 162
Since $\log_a b = c$ if $a^c = b$, the equation can be written as $2^2 = \log_3 \log_4 2^n$ Now use the definition again: $3^{2^2} = \log_4 2^n \rightarrow 81 = \log_4 2^n \rightarrow 4^{81} = 2^n$ Therefore, $n = 2 * 81 = 162$

23. In May, the price of a pair of jeans was 200% of its wholesale cost. In June, the price was reduced by 15%. After another 20% discount in July, the jeans cost \$40.80 . What was the wholesale cost of the jeans?
Answer: \$30.00
Let x be the number of dollars in the wholesale cost of the jeans. Then $0.80*0.85*2.00*x = 40.80$ and $x = 40.80$.

24. Find all ordered pairs of positive integers (a, b) such that $a < b$ and $\sqrt{10 + \sqrt{84}} = \sqrt{a} + \sqrt{b}$
Answer: (3,7)
Square both sides to get $10 + 2\sqrt{21} = a + b + 2\sqrt{ab}$, so $a + b - 10 = 2(\sqrt{21} - \sqrt{ab})$. ab is an integer so $\sqrt{21} - \sqrt{ab}$ is integral only if $ab = 21$ [in general, if x, y, z are positive integers such that x, y have no common factor and $\sqrt{x} - \sqrt{y} = z$, then $x = y = 1$] $a + b = 10$ so $a = 3, b = 7$.

25. Find the sum of the first 50 terms in the arithmetic sequence $2, 6, 10, 14, \ldots$
Answer: 5000
The rule for the sequence is that term n equals $4n - 2$. The 50^{th} term of the sequence is thus $4(50) - 2 = 198$. The first term plus the 50^{th} term is $198 + 2 = 200$. The second term plus the 49^{th} term is $194 + 6 = 200$. Notice that there are 25 pairs of numbers, each of which add to 200. Thus, the sum of the first 50 terms of the sequence is $25(200) = 5000$.

26. Trapezoid $ABCD$ is such that $AB = 43$ cm is parallel to $CD = 207$ cm. $m\angle ADC = 45$ degrees. If the area of the trapezoid is 10500 cm^2, how long is line segment BC, in cm?
Answer: 116
From A draw a perpendicular AH to DC and from B draw a perpendicular BG to DC. Of course, H and G are on DC. It is clear that $10500 = (43 + 207) * AH/2$. Hence $AH = 2 * 10500/(43 + 207) = 84$ and $BC = \sqrt{84^2 + 80^2} = 116$.

27. The Bergen County Academies is hosting a school-wide Super Smash Brothers tournament. Twenty-four students signed up for the tournament. Each player plays one other player in each round, and the winner advances to the next round. Elimination games continue until there is one winner. How many games must be played to find the winner?
Answer: 23
In each game, exactly one player is eliminated from the tournament. There are 24 players, of which 23 must lose (there is one winner). Thus, 23 games must be played.

28. What is the smallest value of k for which $4x^2 + kx - 14x + 25$ is a perfect square for all integer x?
Answer: -6
$4x^2 + (k - 14)x + 25$ is a perfect square if and only if the discriminant $b^2 - 4ac$ is zero. In this case, the discriminant is $(k - 14)^2 - 4 * 4 * 25 = (k - 14)^2 - 20^2$. Hence, $k - 14 = \pm 20 \Rightarrow k = 14 \pm 20 = -6, 34$. The smaller of the two values is -6.

29. At the science fair, Veena presents her project every 15 minutes. She gets 1 visitor at $1:00$. This person is so enthralled, he tells 2 different people to see it at her next presentation. Each of these people has the same reaction, and each tell 2 different people to see it at the next presentation. If this pattern continues, with every person who sees the project making 2 other people see it the next time it is presented, and no one else discovers it, how many people will be present at her last presentation, at $2:30$?
Answer: 64
If a_n is the number of people at the nth presentation, $a_{n+1} = 2a_n$ so $a_n = 2^{n-1}$. 2:30 - 1:00 = 1.5 hours, so there will be 7 presentations and $2^{7-6} = 64$ people.

30. If two distinct numbers are selected at random from the first 13 prime numbers, what is the probability that their sum is even?
Answer: $\frac{11}{13}$
If the sum of two numbers is even, both numbers must be even or both numbers must be odd. Except for 2, all of the prime numbers are odd. Hence the probability that both numbers are even is 0. The probability that both numbers are odd is $\frac{12}{13} * \frac{11}{12} = \frac{11}{13}$.

31. Two people stand back to back next to the rails in a small railway station. As the head of the express train that passes the station reaches them, they start to walk away from each other parallel to the rails. As the tail of the train reaches each of them, they stop, having walked 30m and 40m respectively. If they both walked at the same constant speed and the train moved at a constant speed, how long was the train?
Answer: 240
Note that 60m of train was behind the forward-walking person when he had walked 30m, and that 60m passed him during the time he walked 10 more meters. So, for every 10m he walked, 60m of train passed him. Since he walked 40m in all, the total length of the train that passed him is 240m, which is the length of the train.

32. Lucy arranges 45 chords of a given circle so as to yield the maximum number of points of intersection of two chords inside the circle. What is the maximum number?

Answer: 990
Each intersection point lies exactly on two chords. So any time we chose 2 chords out of 45, we have a point of intersection. Hence the maximum number of intersction points formed is: $45!/(2! * 45!) = 45 * (45 - 1)/2 = 990$.

33. An absentminded bank teller switches the dollars and cents when he cashed a check for Hernando, giving him dollars instead of cents, and cents instead of dollars. After buying a five cent newspaper, Hernando discovered he had left exactly twice as much as his original check. What was the amount of the check?
Answer: $31.63
This problem becomes a question of evaluating a system of equations: $100y+x-5 = 2(100x+y)$ and $y = (199x + 5)/98$. We can rewrite the second equation as $y = 2x + (3x + 5)/98$. So the trick becomes finding an integer solution to $3x + 5 = 98$ or $3x + 5 = 2 * 98$ or $3x + 5 = 3 * 98$. It turns out $3x + 5 = 98$ has an integer solution when $x = 31$, and the other equations don't have integer solutions. When $x = 31$, $y = 63$, so the original check was for $31.63

34. If the integer $(5^3) * (3^b)$ is divisble by only 20 positive integers, then b is equal to:
Answer: 4
A divisor of $(5^3) * (3^b)$ is of the form $(5^s) * (3^t)$ where $0 \le s \le 3$ and $0 \le t \le b$. Hence the number of divisors is $(3 + 1) \times (b + 1) = 20$. Solving for b, gives $b = 4$.

35. Let a, b and c represent the lengths of the sides of a right triangle with $b < c < a$. If $L = 8a^2 + 20b^2 + 20c^2$, find the value of $L/(b^2 + c^2)$.
Answer: 28
It is clear that $a^2 = b^2+c^2$. Hence $L = (8+20)*(b^2+c^2)$. Therefore $L/(b^2+c^2) = (8+20) = 28$.

36. A convex polygon has n sides and $15n$ diagonals. Find the value of n.
Answer: 33
The number of diagonals in a convex n-sided polygon is $d = n * \frac{(n-3)}{2}$. So $n\frac{(n-3)}{2} = 15 * n$ $\rightarrow n - 3 = 2 * 15 \rightarrow n = 2 * 15 + 3 = 33$.

37. In the geometric sequence $1, 2, 4, 8, ...,$ the nth term is 1024. Find n?
Answer: 11
The first term is 1 and the ratio is 2, so the eleventh term is $1 * 2^{10} = 1024$.

38. What is the surface area of a cube with side length 1.5?
Answer: 13.5
The surface area of a cube is equal to 6 times the area of a single side because all 6 sides have the same area. The area of a single side is $1.5*1.5 = 2.25$. Hence the surface area is $2.25*6 = 13.5$.

39. The measure of the supplement of an angle is 4 times the measure of its complement. Find the sum, in degrees, of the measure of the angle.
Answer: 60
Let x° be the angle's measure. Then $(180 - x)^\circ$ is the supplement's measure, $(90 - x)^\circ$ is the complement's measure. $180 - x = 4(90 - x) \Rightarrow x = 60^\circ$

40. How many terminating zeroes does the number 125! have?
Answer: 31
There are many more factors of 2 than factors of 5 in 125!. Thus, the number of $2 - 5$ pairs equals the number of factors of 5 in 125!. Each multiple of 5 has one factor of 5, so there are $125/5 = 25$ factors of 5 there. Each multiple of $5^2 = 25$ has an additional factor of 5; there are $125/25 = 5$ more factors of 5 there. Finally, $5^3 = 125$ has an additional multiple of 5. So, 125! has $25 + 5 + 1 = 31$ factors of 5, so there are $312 - 5$ pairs. Thus, 125! has 31 terminating zeros.

41. Solve for $x : \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \frac{1}{10^5} + \frac{1}{10^6} + \frac{1}{10^7} + \frac{1}{10^8} = \frac{x}{10^8}$.
Answer: 11111111
Multiply both sides of $\frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \frac{1}{10^5} + \frac{1}{10^6} + \frac{1}{10^7} + \frac{1}{10^8} = \frac{x}{10^8}$ by 10^8 to obtain $1 + 10^1 + 10^2 + 10^3 + 10^4 + 10^5 + 10^6 + 10^7 = x$. Hence $x = 11111111$

42. Consider a rectangle $ABCD$ where $AB = 6cm$, and $BC = 9$ cm. Let M be a point on the segment AB. In cm^2, what is the area of $\triangle MCD$?
Answer: 27
$\triangle MCD$ has height 9 cm and base 6 cm. Its area is thus $\frac{9*6}{2} = 27$.

43. What is the greatest number of right triangular sections, each with base = 3 inches and height = 5 inches, that can be cut from a rectangular piece of paper measuring 55 inches by 21 inches?
Answer: 154
Since we are cutting from a rectangular sheet of paper, the number of right triangular sections is twice the number of rectangles of sides 3 and 5. To find the greatest number of rectangles, one has to break up 55 inches into 11 pieces of 5 inches each and 21 inches in 7 pieces of 3 inches each. Hence the answer is $2 * (7 * 11) = 154$ triangular sections.

44. When each side of a square is increased by 2 feet, the area is increased by 24 square feet. By how many feet does each side of the original square have to be decreased in order to decrease the area of the original square by 24 square feet?
Answer: 4
Let x be the original side of the square. Then $(x+2)^2 - x^2 = 24$ Solving for x yields $x = 5$. Let $y > 0$ be the new side length. Then, $5^2 - (y)^2 = 24$ so $y = 1$ and the difference is $5 - 1 = 4$

45. How many points, no three of which are collinear, determine 91 lines?
Answer: 14
Since no three points are collinear, there is a unique line for every pair of points. Thus, letting x represent the number of points, we find: $\binom{x}{2} = 91 \Rightarrow (x-1) * \frac{x}{2} = 91 \Rightarrow x = 14$.

46. $\frac{\sin 10 \cos 10 \tan 10 \cot 10 \sec 10 \csc 10}{\sin 20 \cos 20 \tan 20 \cot 20 \sec 20 \csc 20} = ?$
Answer: 1
$\sin x \csc x = 1, \cos x \sec x = 1$, and $\tan x \cot x = 1$, so $\frac{\sin 10 \cos 10 \tan 10 \cot 10 \sec 10 \csc 10}{\sin 20 \cos 20 \tan 20 \cot 20 \sec 20 \csc 20} = 1$.

47. In the coordinate plane, the graph of the function f is the point $(3, -4)$. What is the distance between the graph of f and the graph of its inverse, f^-1?
Answer: $7\sqrt{2}$
The graph of f^{-1} consists of the single point $(-4, 3)$. Hence the distance between the points $(3, -4)$ and $(-4, 3)$ is $\sqrt{(-4-3)^2 + (3-(-4)^2)} = \sqrt{98} = 7\sqrt{2}$

48. A sequence of number $a_1, a_2, a_3, a_4, \ldots$ is defined by $a_1 = 7, a_2 = -6$ and $a_n = a_{n-1} - a_{n-2}$. What is the sum of the first 1587 terms of the sequence?
Answer: -12
The sequence repeats itself every 6 terms: $a_1 = 7, a_2 = -6, a_3 = -6 - 7 = -13, a_4 = -13 - (-6) = -7, a_5 = -7-(-13) = 6, a_6 = 6-(-7) = 13, a_7 = 13-6 = 7$, etc. The sum returns every 6 terms: $s(n+6) = s(n)+7-6-13-7+6+13 = s(n)$. $s(1587) = s(1581) = \ldots = s(3) = -12$.

49. $f(x) = x^2 + x + 11$. x is chosen from $\{1, 2, \ldots, 9\}$. What is the probability that $f(x)$ is prime?
Answer: 1 or 100%
$f(x)$ is in $\{13, 17, 23, 31, 41, 53, 67, 83, 101\}$. Each term is prime, so the probability is $\frac{9}{9} = 1$.

50. $1/2 + 1/6 + 1/12 + 1/20 + \ldots + 1/4830 =?$
Answer: 69/70
Notice that $1/2 + 1/6 + 1/12 + 1/20 + ... + 1/4830 = 1/(1 * 2) + 1/(2 * 3) + 1/(3 * 4) + 1/(4 * 5) + \ldots + 1/(69 * (69 + 1))$. Notice that $1/(a * (a + 1)) = 1/a - 1/(a + 1)$. Applying this to the above sum gives: $(1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \ldots + (1/69 - 1/(69 + 1))$. The above sum can be reduced to $1 - 1/(69 + 1) = 69/70$.