4th Grade Competition Solutions

Bergen County Academies Math Competition
19 October 2008

1. Before taking the AMC, a student notices that he has two bags of Doritos and one bag of Skittles on his desk. Each bag of Doritos is 130 calories, and each bag of Skittles is 60 calories. If the student eats both bags of Doritos and eats the Skittles, then how many calories will he have eaten?
Answer: 320
$2 \times 130 + 1 \times 60 = 260 + 60 = 320$.

2. A section of seats at Beaver Stadium has 80 rows. If each row has 50 seats, then how many seats are in the entire section of seats?
Answer: 4000
The total number of seats is $80 \times 50 = 4000$.

3. Find $35 \div 5$.
Answer: 7
$35 \div 5 = 7$.

4. David Rush is in a hurry to get to class. His dorm at Philips Exeter is 600 meters away from his first class. If he runs to class at a speed of three meters per second, how many seconds does it take him to get to class?
Answer: 200
Time = distance ÷ rate. Thus, it takes David (600 m) ÷ (3 m/s) = 200 seconds to get to class.

5. $\triangle ABC$ has sides of lengths 4, 5, and 6. Find the perimeter of $\triangle ABC$.
Answer: 15
The perimeter of a triangle is equal to the sum of the lengths of its sides, so the answer is $4+5+6=15$.

6. Find $37 \times 108 + 37 \times 92$.
Answer: 7400
$37 \times 108 + 37 \times 92 = 3996 + 3404 = 7400$.
OR
By the distributive law, $37 \times 108 + 37 \times 92 = 37 \times (108+92) = 37 \times 200 = 7400$.

7. $4+44+444+4444+44444 = ?$
Answer: 49380
$4+44+444+4444+44444 = 49380$.
OR
By the distributive law, $4+44+444+4444+44444 = 4 \times (1+11+111+1111+11111) = 4 \times 12345 = 49380$.

8. Suppose that $a@b = 17a^2 - 13b^3$. Find $3@2$.
Answer: 49
By definition, $3@2 = 17 \times 3^2 - 13 \times 2^3 = 17 \times 9 - 13 \times 8 = 153 - 104 = 49$.

9. Compute $3 + 4 \times (20-10) \div 2$.
Answer: 23
$3 + 4 \times (20-10) \div 2 = 3 + 4 \times 10 \div 2 = 3 + 40 \div 2 = 3 + 20 = 23$.

10. Brian Basham bashes rocks for exercise. On the first day, he bashed one rock. The next day he bashed three rocks, on the third day he bashed five rocks, and so on. How many total rocks did he bash after five days of exercise?
Answer: 25

Brian bashed a total of $1+3+5+7+9 = 25$ rocks. (Is it a coincidence that this answer is a perfect square?)

11. A dog runs in a circular path of radius 16 feet. If the dog completes exactly one run around the circle, then how many feet will the dog have travelled? Round your answer to the nearest foot.
Answer: 100.5
The circumference of a circle with radius of length r is $2\pi r$. So, our answer is $2\pi \times 16 = 32\pi \approx 32 \times 3.14 = 100.48 \approx 100.5$.

12. Bob is thinking of a number that leaves a remainder of 5 when divided by 12. What remainder does Bob's number leave when divided by 4?
Answer: 1
Bob's number is equal to $12n+5$, for some number n. So, Bob's number is also equal to $4(3n+1)+1$, and leaves a remainder of 1 when divided by 4.

13. A trapezoid has three angles that measure 82, 47, and 98 degrees. What is the degree measure of the fourth angle?
Answer: 133
By drawing a diagonal of any quadrilateral (and thus separating the quadrilateral into two triangles), it is clear that the angles in any quadrilateral sum to 360^o. Then the fourth angle in this trapezoid has measure $360-(82+47+98) = 360-227 = 133^o$.

14. How many numbers less than 1000 are both perfect cubes *and* perfect squares?
Answer: 4
Any number that is both a perfect cube and a perfect square has a prime factorization with exponents that are all divisible by both 2 and 3. So, they are also all divisible by 6, meaning the number is equal to some integer to the sixth power. Since $3^6 = 729 < 1000 < 4096 = 4^6$, there are 4 perfect sixth powers less than 1000. (Remember 0!!!)

15. Find the perimeter of a right triangle with legs of lengths 6 and 8.
Answer: 24
The hypotenuse of this right triangle is equal to $\sqrt{6^2+8^2} = \sqrt{36+64} = \sqrt{100} = 10$. The perimeter is then $6+8+10 = 24$.

16. A football team has 105 members. If 45 of these players play at least one play in a certain game, then what fraction of the team played at least one play that game?
Answer: $\frac{3}{7}$
$\frac{45}{105} = \frac{9}{21} = \frac{3}{7}$. Since 3 and 7 share no common factors other than 1, this fraction is fully reduced.

17. What is the length of a side of a square with area 121?
Answer: 11
The area of a square of side length s is s^2. Since $121 = 11^2$, the square has a side of length 11.

18. Josh and Kun-Soo have 2008 Frosted Flakes each. Josh gives Kun-Soo 550 flakes, but Kun-Soo gives 692 flakes back to Josh. How many flakes does Josh have now?
Answer: 2150
After the first exchange, Josh has $2008-550 = 1458$ flakes. After the second exchange, Josh has $1458+692 = 2150$ flakes.

19. At math team, three bags of chips is equal to a can of soda and 7 boxes of Nerds are equal to three bags of chips. How many cans of soda are 35 boxes of Nerds equal to?
Answer: 5
35 boxes of Nerds are equal to $\frac{3}{7} \times 35 = 3 \times 5 = 15$ bags of chips, which are equal to $\frac{1}{3} \times 15 = 5$ cans of soda. The

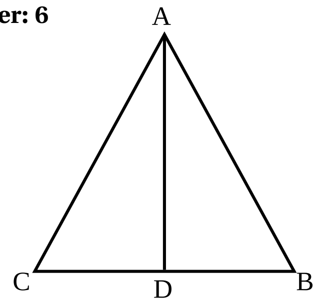
calculation can be done directly: $35 \times \frac{3}{7} \times \frac{1}{3} = 35 \div 7 = 5$.

20. Emily, John, Ben, and Craig are playing a game of hearts. They are each dealt 13 cards from a standard 52 card deck. What is the probability that Emily gets the queen of spades?
Answer: $\frac{1}{4}$
Each of the four players has an equal chance of getting the queen of spades, so Emily has a $\frac{1}{4}$ chance of getting it.

21. How many prime numbers are less than 20?
Answer: 8
2, 3, 5, 7, 11, 13, 17, and 19 are the eight prime numbers less than 20.

22. Find the perimeter of a rectangle that has area 20 and one side of length 2.
Answer: 24
The other side of the rectangle must have length $20 \div 2 = 10$. The perimeter of this rectangle is $2(10+2) = 24$.

23. A 30-60-90 right triangle has a hypotenuse of length 12. Find the length of the shorter leg.
Answer: 6



Consider an equilateral triangle ABC with median $\overline{AD}$ drawn, as shown. Because of the symmetry in the drawing, $\angle ABD = 60^o$ and $\angle BAD = 30^0$. Then $\triangle ABD$ is a 30-60-90 triangle, and the length of $\overline{BD}$ is one-half that of $\overline{AB}$, or 6.

24. A is a digit (a number from 0-9). Find the sum of all A that make the number $612663A$ even.
Answer: 20
Each of the digits 0, 2, 4, 6, and 8 would make the number 612663A even. Our sum is $0+2+4+6+8 = 20$.

25. What is the 59^{th} term of the sequence $A, B, C, D, A, B, C, D, A, B, C, D, A, B, C, D, ...$?
Answer: C
The sequence is a repetition of the four letters A, B, C, D. Since 59 leaves a remainder of 3 when divided by 4, the 59^{th} letter of the sequence will be the third letter of the sequence A, B, C, D, which is C.

26. Compute the volume of a cube with side length 13.
Answer: 2197
The volume of the cube will be $13^3 = 13 \times 13 \times 13 = 169 \times 13 = 2197$.

27. $1+0+1+1+1+2+1+3+1+4+1+5+1+6+1+7+1+8+1+9 = ?$
Answer: 55
The addition could be done directly, but is made slightly shorter if consecutive pairs of digits are paired. Then the sum becomes $1+2+3+\ldots 9+10 = \frac{10 \times 11}{2} = 55$.

28. How many $2 \times 2 \times 2$ blocks does it take to completely fill a $10 \times 12 \times 14$ box?
Answer: 210
The volume of a $10 \times 12 \times 14$ box is $10 \times 12 \times 14 = 1680$, while the volume of each $2 \times 2 \times 2$ block is $2^3 = 8$. Then

$1680 \div 8 = 210$ small blocks will fit in the box.

29. What is the area of a circle with radius 3, rounded to the nearest tenth? Answers in terms of π are not accepted.
Answer: 28.3
The area of a circle with radius of length r is πr^2. When $r = 3$, this is $3^2 \times \pi = 9\pi \approx 9 \times 3.14 = 28.26 \approx 28.3$.

30. Find the area of an isosceles triangle with sides of length 4, 8, and 8.
Answer: $4\sqrt{15}$
Draw the altitude from the vertex of the triangle to the base. By the Pythagorean Theorem, the height of the isosceles triangle is $\sqrt{8^2 - 2^2} = \sqrt{64-4} = \sqrt{60} = 2\sqrt{15}$. So the area of the triangle is $\frac{1}{2} \times 4 \times 2\sqrt{15} = 4\sqrt{15}$.

31. Hannah is cramming for the SAT, and needs to memorize 150 words. She can cram 50 words each morning, but will forget 40 of them after that night. After how many mornings of studying will Hannah be able to write the correct definition of all 150 words?
Answer: 11
After two mornings of study, Hannah will have memorized $50+50-40 = 60$ words. Each morning hereafter, Hannah will have memorized $50-40 = 10$ more words than she will have had memorized the day before. Thus, Hannah will finish all 150 words after $2+90 \div 10 = 11$ mornings of study.

32. The *semiperimeter* of a triangle is one-half of the perimeter. The inradius (r), area (K), and semiperimeter (s) of a triangle are related by the equation $K = rs$. Find the inradius of a right triangle with sides of lengths 7, 24, and 25.
Answer: 3
Since $K = rs$, $r = \frac{K}{s}$. $K = \frac{1}{2} \times 7 \times 24 = 7 \times 12$, and $s = \frac{7+24+25}{2} = 28 = 7 \times 4$. Then $r = \frac{7 \times 12}{7 \times 4} = \frac{12}{4} = 3$.

33. If Russell has a colony of ants that double in population every half hour, how many ants will he have at the end of four hours if he starts with 2 ants?
Answer: 512
The population of a colony with a beginning population of a ants after a time of h half hours is $a \times 2^h$. So our answer is $2 \times 2^8 = 2^9 = 512$.

34. Compute $1^2 + 2^2 + 3^2 + 4^2 + 5^2$.
Answer: 55
$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1+4+9+16+25 = 55$. (There is a formula for the sum S_n of the first n squares: $S_n = \frac{n(n+1)(2n+1)}{6}$.)

35. The formula for converting Fahrenheit temperatures to Celsius temperatures is $C = \frac{5}{9}(F-32)$. Convert 77^oF to degrees Celsius.
Answer: 25
Using the formula, our answer is $\frac{5}{9}(F-32) = \frac{5}{9}(77-32) = \frac{5}{9}(45) = 25^o$

36. Austin runs an animal hospital that takes care of cats and birds. The cats here have four legs and no tails while the birds have two legs and two tails each. Victoria, the hospital inspector, walks in one day and counts fifteen heads, 46 legs, and 14 tails. How many cats were there?
Answer: 8
Since the cats have no tails we know that there are $14 \div 2 = 7$ birds. Thus, the cats contribute $46 - 7 \times 2 = 32$ legs. This means there must be $32 \div 4 = 8$ cats. Notice that the number of heads is unneeded information.

37. Nikhil is writing the page numbers for a book that has 37 pages. The first page is page 1. How many digits does Nikhil have to write to accomplish this task?
Answer: 65

There are 9 one-digit page numbers, so the one-digit page numbers contribute 9 digits. There are $37-10+1=28$ two-digit page numbers, so the two-digit page numbers contribute $28\times 2=56$ digits. In total, Nikhil must write $56+9=65$ digits.

38. What is the sum of the interior angles of a hexagon, a figure with 6 sides?
Answer: 720^o
If we draw three non-intersecting diagonals in the hexagon, it will be divided into 4 triangles. Since each triangle contributes 180^o, the hexagon must have $4\times 180^o=720^o$ degrees in total.

39. Tanya wants to arrange her dresses in a line. She has five dresses: 2 red, 1 green, 1 yellow, and 1 blue. She does not want the two red dresses to be next to each other. In how many ways can she line up her dresses?
Answer: 36
In all, there are $\frac{5!}{2!}=\frac{120}{2}=60$ ways to arrange the five dresses in a line, not considering the condition that the two red dresses must not be next to each other. By keeping the two red dresses together and counting them as one unit, one can see that there are $4!=24$ arrangements of the dresses in which the two red dresses are next to each other. Then the number of ways Tanya can line up her dresses in the manner described is $60-24=36$.

40. It is pitch black, and James is picking socks out of a drawer containing 10 red socks, 10 blue socks, 10 yellow socks, and 10 white socks. James cannot see the color of the socks that he is picking out of the drawer. How many socks must James take out of the drawer to be sure that he has taken out 4 pairs of socks? A pair of socks is two socks that are the same color. Each sock can only be counted in one pair.
Answer: 11
After 10 socks, James may not have picked four pairs – he could have picked 7 red, 1 blue, 1 yellow, and 1 white sock, for example. After 11 socks, however, James has either picked an odd number of socks of only one color or an odd number of socks of exactly three colors. Ignoring these 'extra' socks, he has either picked 10 paired socks or 8 paired socks. In either case, James has at least 4 pairs of socks.

41. If $n!=n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$, then find $\frac{6!}{4!}$.
Answer: 30
$\frac{6!}{4!}=\frac{6\times 5\times 4\times 3\times 2\times 1}{4\times 3\times 2\times 1}=6\times 5=30.$

42. In how many ways can 10 be written as a sum of three (not necessarily distinct) positive whole numbers? The order of the numbers does not matter.
Answer: 8
Consider the greatest of the three numbers; it must be at least 4 because if it were 3, the sum of the three numbers would be at most $3+3+3=9<10$. If it is 4, the other two numbers could be 2 and 4 or 3 and 3. If it is 5, the other two numbers could be 1 and 4 or 2 and 3. If it is 6, the other two numbers could be 1 and 3 or 2 and 2. If it is 7, the other two numbers must be 1 and 2. If it is 8, the other two numbers must be 1 and 1. In all, there are $2+2+2+1+1=8$ possible sums.

43. If I flip a coin that has one side heads and one side tails three times, what is the probability that I get heads, tails, heads, in that order?
Answer: $\frac{1}{8}$
The probability of each flip occurring is $\frac{1}{2}$. So, the probability of flipping the sequence heads, tails, heads is $\left(\frac{1}{2}\right)^3=\frac{1}{8}$.

44. Find the sum of the first 30 terms of the arithmetic sequence 3, 7, 11, …
Answer: 1830
The n^{th} term of the sequence is given by the formula $4n-1$. Term n and term $31-n$ of the sequence sum to $(4n-1)+(4(31-n)-1)=4n-1+124-4n-1=124-2=122$. There are 15 such pairs of terms among the first 30 terms of the sequence; thus, the sum of the first 30 terms of the sequence is $15\times 122=1830$.

45. Convert 111_6 – a number written in base 6 – to base 10.
Answer: 43
$111_6 = 1 \times 6^2 + 1 \times 6^1 + 1 \times 6^0 = 1 \times 36 + 1 \times 6 + 1 \times 1 = 43$.

46. Compute the volume of a cube that has a surface area of 216.
Answer: 216
The surface area of a cube of side length s is $6s^2$ since a cube has six square faces. When $6s^2 = 216 = 6 \times 36 = 6 \times 6^2$, the side of the cube must be equal to 6. The volume of a cube with side length 6 is $6^3 = 216$.

47. How many numbers between 1 and 101 are multiples of 3 or 4?
Answer: 50
There are 33 multiples of 3 between 1 and 101. There are 25 multiples of 4 between 1 and 101. Summing these, there are $33 + 25 = 58$ multiples of 3 or 4 between 1 and 101. But since multiples of 12 are multiples of both 3 and 4, we have counted them twice!! Since there are 8 multiples of 12 between 1 and 101, our answer is $58 - 8 = 50$.

48. A palindrome is a number that is read the same way forwards as it is backwards. For example, the numbers 14541 and 1221 are palindromes, but the number 132531 is not. How many three-digit numbers are palindromes? (A three-digit number cannot start with the digit 0.)
Answer: 90
A three-digit palindrome is determined by its first two digits (since the third digit must be equal to the first). There are 9 choices for the first digit (1 through 9) and 10 choices for the second digit (0 through 9). Thus, there are $9 \times 10 = 90$ three-digit palindromes.

49. Find a number x besides 0 that satisfies $|x - |3x|| < 1$.
Answer: $-\frac{1}{4} < x < \frac{1}{2}, x \neq 0$
We will solve for x and find all values that satisfy the inequality. $|x - |3x|| < 1 \Rightarrow -1 < x - |3x| < 1$. If $x \geq 0$, this simplifies to $-1 < -2x < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2} \Rightarrow 0 \leq x < \frac{1}{2}$ (since we said $x \geq 0$). If $x < 0$, the original inequality simplifies to $-1 < x - (-3x) < 1 \Rightarrow -1 < 4x < 1 \Rightarrow -\frac{1}{4} < x < \frac{1}{4} \Rightarrow -\frac{1}{4} < x < 0$ (since we said $x < 0$). Thus, any x strictly between $-\frac{1}{4}$ and $\frac{1}{2}$ other than 0 is a correct answer.

50. A positive whole number is called relatively prime to another positive whole number if it shares no factors other than 1 with the second number. How many numbers less than 38 are relatively prime to 38?
Answer: 18
Any number that is not divisible by 2 or 19 is relatively prime to 38. So, every odd number from 1 to 37 inclusive, besides 19, is relatively prime to 38. There are $38 \div 2 - 1 = 18$ of these.