7th Annual Bergen County Academies Math Competition

Seventh Grade

Sunday, 18 October 2009

1 Rules

1. You may use space on your test paper and additional scrap paper to do work. Your answers must be written on the answer sheet. We will not look at answers written on your test paper.
2. Each problem has only one answer. If you put more than one answer for a problem, you will be marked wrong. When changing an answer, be sure to erase or cross out completely.
3. Write legibly. If the graders cannot read your answer, it will be marked incorrect.
4. Fractions should be written in lowest terms. For example, if the answer is $\frac{1}{2}$, then $\frac{2}{4}$ will not be accepted although the two fractions are numerically equal.
5. All other answers should be written in simplest form.
6. If a unit is indicated in the problem, the answer must be given in that unit. For instance, if the problem asks for the answer in hours, you cannot give your answer in minutes. Furthermore, you don't need to write the unit, as the graders will assume your answer is in the units asked for in the problem.
7. There is no penalty for guessing.
8. Ties will be broken based on the number of correct responses to the last ten questions. If a tie remains, then the correct responses to the last five questions will break the tie.
9. We will announce how much time is remaining often during the test.

2 Contest

1. Sam, who is 5.5ft tall, is standing near Boston Market, which is 17.5ft tall. He notices that his shadow is 10ft long. In feet, how long is Boston Market's shadow?

2. How many positive integers less than 50 have exactly two prime factors?

3. John was walking home when he realized that he could save 15 minutes of walking time if he walked directly home instead of walking first 90 meters to the east and then 120 meters south. What is his walking speed in meters per minute?

4. Suppose I have a standard deck of 52 cards. What is the probability that when I draw a card, it is a heart or an ace?

5. Point B lies on the same plane as line $\overline{AC}$ such that B is nine units away from C and eleven units away from A. If the length of $\overline{AC}$ is an integer, let L be the longest possible length of $\overline{AC}$ and l be the shortest possible length of $\overline{AC}$. What is $L - l + 1$?

6. The ratio of a to b is $5:1$ and the ratio of b to c is $\frac{3}{2}$. What is $a:c$?

7. What is the maximum possible number of intersection points among three circles and three lines, all arranged in the same plane?

8. What is the next term in the following sequence?
$$1, 2, -2, 4, 4, 6, -8, 8, 16, 10, ...$$

9. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x and $\lceil x \rceil$ be the least integer greater than or equal to x. Evaluate $10\lceil \lfloor -1.7 \rfloor + 3.3 \rceil - 3$.

10. In $\triangle ABC$, $\overline{AB} = 5$, $\overline{AC} = 6$, and $\overline{BC} = 7$. Find the area of $\triangle ABC$.

11. Alex, Brian, Charles, David, and Ed competed in a race, in which no one tied. Given that Alex beat Charles, Brian beat David and Ed, and Charles beat David, how many different orders of finishers could there have been?

12. Let circle A have radius R, where R is a positive integer between 1 and 10 inclusive. We also know that R has four distinct positive integer divisors. Find the sum of all possible areas of A.

13. A six-sided die is rolled four times. What is the probability that two sixes are rolled consecutively?

14. When the elements in the set $\left\{\frac{\pi}{2}, \frac{\pi}{2} \cdot \frac{\pi}{3}, \left(\frac{\pi}{3}\right)^2 \cdot \frac{\pi}{4}\right\}$ are simplified to lowest terms, what is the maximum value?

15. Let the number n be your answer for this problem. What is $6 - \frac{9}{n}$?

16. Express $0.123\overline{45}$ as a fraction.

17. There are four pandas being sent off to four zoos, with each panda going to a random zoo. What is the probability that one zoo gets all four pandas?

18. Kamran got an iPod touch and wanted to measure its battery life. After five runs, he recorded the following times in hours: 22.7, 19.125, 14.3, 23, and 20.875. What was the average battery life in hours?

19. Matt randomly picks two numbers (not necessarily distinct) out of the set $\{\pi, 2, \sqrt{2}, 3.5\}$. Let p be the probability that the sum of the two numbers is an integer, and let q be the probability that the product of the two numbers is an integer. Find $p+q$.

20. If one side of a triangle is 50 cm long and the another side is 60 cm long, how many possible integer side lengths are there for the last side?

21. Suppose $A^2+B^2+C^2=4$ and $AB+BC+CA=3$. Find $(A+B+C)^2$.

22. Mr. Holbrook runs around the track. He ran the first lap at 30 yards per minute. How fast must he run the next lap to increase his average speed to 45 yards per minute?

23. What is the equation of the line formed when the line $x+2y=5$ is reflected across the x-axis and then the y-axis? Write your answer in the form $Ax+By=C$, where A, B, and C are integers and A is positive.

24. Ashley likes to shop and is about to buy a pair of pants when she realizes that she is 12 dollars short of the price. She asks her friend Janet, who started out with .25 times the amount of money she did, to lend her 12 dollars. Janet, who has not spent any money, agrees and is left with 20 dollars. How much money did the two have in the beginning of their shopping trip combined?

25. How many different ways are there to arrange the characters of the word "Puffball?"

26. Let $A=(3,4)$ and $B=(11,10)$. Rotate B 30^o counterclockwise around A to get C. What is the area of $\triangle ABC$?

27. If a mile is 5280 feet, what is the area, in square feet, of a 0.0625 square miles?

28. Mr. Pinyan has two red balls, three blue balls, and four white balls. All balls of the same color are indistinguishable. How many ways are there of arranging the nine balls in a circle?

29. If the perimeter of a rectangle is 28 and the length of its diagonal is 10, find its area.

30. Suppose $P(x)=x(x+1)(x+2)(x+3)(x+4)(x+5)(x+6)$. Find the greatest common factor of $P(1)$, $P(2)$, $P(3)$, ...

31. Find the integer x that satisfies $x^3-3x^2+3x+124=0$.

32. When evaluated, the difference $2^{10}-2^9$ can be expressed in the form a^b, where a and b are integers and a is a prime number. If i is the square root of -1, what is i^{-b^a}?

33. Find the number of positive integer solutions to the equation $2x+3y=101$.

34. In Zhuland, quarters have radius 5 cm and dimes have radius 1 cm. If I drop a dime onto a quarter such that its center lies in the interior of the quarter (with every point in the interior having equal probability of being chosen), find the probability that the dime lies entirely within the quarter.

35. If $f(x)=x^3+ax^2+bx+c$, $f(1)=34$, and $f(4)=100$, find $f(3)-f(2)$.

36. While reading a 123-paged book, Amanda decided to count the page numbers. How many times did the digit 3 show up in the page numbers?

37. Consider the unit sphere (a sphere with radius 1) centered at the origin. Find the shortest possible distance between a point on the unit sphere and the point (3, 4, 5).

38. If a circle intersects the x-axis at points $(6,0)$, and $(8,0)$, and goes through the point $(3,2)$, find the radius of the circle.

39. Michael, Alex, and Kelvin are writing a fantasy book for JP. Working at a constant rate, they can write the entire book in 196 days, 98 days, and 49 days, respectively. How many days will it take for them to finish writing the book if they work together?

40. At which point(s) do the parabola $y = x^2 - 2x + 1$ and the line $2x + y = 1$ intersect?

41. Let $f(n)$ be the remainder when n^3 is divided by 7. What is $f(f(f(3)))$?

42. A blindfolded Brian is trying to pin the tail on the donkey. He knows that the donkey, a circle with radius 2ft, is on a 5ft x 5ft poster board. It is guaranteed that tail will land on the poster board. Let p be the probability that he pins the tail on the donkey and q be the probability that he misses the donkey. What is $p + q$?

43. How many integers from 1 to 1000 are multiples of 12 but not 15?

44. Paul built a very mathematical snowman. The snowman consisted of three spheres whose radii formed an increasing geometric sequence with common ratio $\dfrac{3}{2}$. If the volume of the top sphere was 171π less than that of the bottom sphere and the radius of the smallest sphere can be written as $a\sqrt[3]{\dfrac{b}{c}}$ where b and c do not contain any perfect cube factors other than 1, what is $a + b + c$?

45. A 68-digit number β, when written in binary (base 2), is made up of sixty-eight 1's. Find $\log_2(1 + \beta)$.

46. How many factors does 84 have?

47. Michael and Jordan are competing in a basketball duel. Mike has a $\frac{2}{3}$ chance of winning a match and Jordan has a $\frac{1}{3}$ chance of winning a match (no ties are allowed). What is the probability that Jordan will win in three or fewer matches if they play five matches?

48. Evaluate $1 + \dfrac{2}{1 + \frac{2}{1 + \frac{2}{1 + \cdots}}}$.

49. The numbers 1-4 are written on a board. At any time, we may pick three numbers a, b, and c on the board and replace them with $\frac{a+b}{2}$, $\frac{b+c}{2}$, and $\frac{c+a}{2}$. If M and m are the greatest and smallest sums of the four numbers that can be attained after doing the operation twice, respectively, then find $m \cdot M$.

50. Find the minimal value of

$$\sqrt{(a-1)^2 + (b-1)^2} + \sqrt{(a-1)^2 + b^2} + \sqrt{a^2 + (b-1)^2} + \sqrt{a^2 + b^2}$$

where a, b are arbitrary real numbers.