7th Annual Bergen County Academies Math Competition

Seventh Grade

Sunday, 18 October 2009

1. Sam, who is 5.5ft tall, is standing near Boston Market, which is 17.5ft tall. He notices that his shadow is 10ft long. In feet, how long is Boston Market's shadow?

   Solution: Since the ratio of an object's height to its shadow is constant, we can equate the ratio of Sam's height to his shadow length with Boston Market's height to its shadow length. $\frac{17.5}{5.5} = \frac{\text{Boston Market's shadow}}{10}$. Boston Market's Shadow $= \boxed{\frac{350}{11} = 31\frac{9}{11} \text{ feet}}$.

2. How many positive integers less than 50 have exactly two prime factors?

   Solution: As 50 is a relatively small number, the easiest way to do this would be merely to enumerate all the numbers. We see that 6, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 33, 34, 35, 38, 39, 40, 44, 45, 46, 48 are the only integers less than 50 satisfying our properties, so there are $\boxed{22}$ integers satisfying our properties.

   Note: If 50 were replaced by a larger number, another much more complicated approach would be desirable. However, since 50 is such a small number, it's much easier to simply list all our candidates out than finding a "smart way" to enumerate them.

3. John was walking home when he realized that he could save 15 minutes of walking time if he walked directly home instead of walking first 90 meters to the east and then 120 meters south. What is his walking speed in meters per minute?

   Solution: Let r be the rate at which John walks in meters per minute. John covers 210 meters in his original route, and therefore takes $\frac{210}{r}$ minutes to arrive home. He covers 150 meters in his revised route, and therefore takes $\frac{150}{r}$ minutes to arrive home. We are given that $\frac{150}{r} + 15 = \frac{210}{r}$. Multiplying both sides by r, we see that $150 + 15r = 210$, i.e., $15r = 60$. Thus, $r = \boxed{4}$.

4. Suppose I have a standard deck of 52 cards. What is the probability that when I draw a card, it is a heart or an ace?

   Solution: Let $P(H)$ and $P(A)$ be the probabilities that a heart is drawn and an ace is drawn, respectively. We see $P(H \text{ or } A)$. The principle of inclusion-exclusion of probability tells us that $P(H \text{ or } A) = P(A) + P(H) - P(A \text{ and } H)$. $P(A) = \frac{1}{13}$ and $P(H) = \frac{1}{4}$. Since there is only one card that is both an ace and a heart, $P(A \text{ and } H) = \frac{1}{52}$. Therefore, $P(H \text{ or } A) = \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{4+13-1}{52} = \boxed{\frac{4}{13}}$.

5. Point B lies on the same plane as line $\overline{AC}$ such that B is nine units away from C and eleven units away from A. If the length of $\overline{AC}$ is an integer, let L be the longest possible length of $\overline{AC}$ and l be the shortest possible length of $\overline{AC}$. What is $L - l + 1$?

   Solution: The three segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ form a triangle, so they have to satisfy the triangle inequality, which states that the sum of any two side lengths must be greater than that of the third side. Therefore, $\overline{AC} + \overline{BC} > \overline{AB}$ and $\overline{AB} + \overline{BC} > \overline{CA}$. Substituting numbers, we have $\overline{AC} + 9 > 11$ and $9 + 11 > \overline{AC}$, which follows that $20 > \overline{AC} > 2$, so $L = 19$ and $l = 3$. As a result, $L - l + 1 = 19 - 3 + 1 = \boxed{17}$.

6. The ratio of a to b is $5 : 1$ and the ratio of b to c is $\dfrac{3}{2}$. What is $a : c$?

   Solution: The problem tells us that $\frac{a}{b} = 5$ and $\frac{b}{c} = \frac{3}{2}$. Multiplying these two equations together, we see that $a : c = \frac{a}{c} = \boxed{\dfrac{15}{2}}$.

7. What is the maximum possible number of intersection points among three circles and three lines, all arranged in the same plane?

   Solution: Just like a three-way Venn diagram, 3 circles will intersect in at most 6 places. The maximum amount of times the first line can intersect the 3 circles is twice for each circle - 6 points. The second line can cross each circle twice and the first line once - 7 more intersections. The third line can cross each circle twice and the first and second lines - 8 more points. $6{+}6{+}7{+}8 = \boxed{\text{27 intersections}}$.

8. What is the next term in the following sequence?

$$1, 2, -2, 4, 4, 6, -8, 8, 16, 10, ...$$

   Solution: Let our sequence be $a_1, a_2, \ldots$. At all the even spots, we have exactly the even numbers. For example, $a_2 = 2$ and $a_4 = 4$. On the other hand, at the odd spots, we have powers of -2: at the first odd spot, we have 1; at the second, we have -2; at the third, we have 4. This continues on. The next term in our sequence has an odd spot in the sequence, and is thus the next power of -2, so our answer is $\boxed{-32}$.

9. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x and $\lceil x \rceil$ be the least integer greater than or equal to x. Evaluate $10\lceil \lfloor -1.7 \rfloor + 3.3 \rceil - 3$.

   Solution: $\lfloor -1.7 \rfloor = -2$, so we wish to find $10\lceil -2 + 3.3 \rceil - 3 = 10\lceil 1.3 \rceil - 3$. Since $\lceil 1.3 \rceil = 2$, we have $10\lceil 1.3 \rceil - 3 = 20 - 3$, giving a final answer of $\boxed{17}$.

10. In $\triangle ABC$, $\overline{AB} = 5$, $\overline{AC} = 6$, and $\overline{BC} = 7$. Find the area of $\triangle ABC$.

    Solution: Solution 1: Let $s = \frac{AB+BC+CA}{2} = \frac{5+6+7}{2} = 9$. By Heron's formula,

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{9 \cdot 4 \cdot 3 \cdot 2} = 6\sqrt{6}$$

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    Solution 2: Let A_1 be the foot of the altitude from A to $\overline{BC}$. We wish to find the length AA_1. Let $x = CA_1$; we have that $BA_1 = 7 - x$. By the Pythagorean theorem, we have $AA_1^2 = AC^2 - A_1C^2 = AB^2 - BA_1^2$, that is, $AA_1^2 = 36 - x^2 = 25 - (7 - x)^2$. Rearranging, $60 = 14x$, so $x = \frac{30}{7}$. We thus have that $AA_1^2 = 36 - \frac{900}{49} = \frac{864}{49}$. It follows that $AA_1 = \sqrt{\frac{864}{49}} = \frac{12\sqrt{6}}{7}$. The area of $\triangle ABC$ is therefore $\frac{AA_1 \cdot BC}{2} = \boxed{6\sqrt{6}}$.

11. Alex, Brian, Charles, David, and Ed competed in a race, in which no one tied. Given that Alex beat Charles, Brian beat David and Ed, and Charles beat David, how many different orders of finishers could there have been?

    Solution: Assign to A, B, C, D, and E the integers 1-5, with the higher integer representing the higher place. The problem tells us that $A > C > D$. We do casework on the location of E. If $A > C > D > E$, then $B > A$, $A > B > C$, or $C > B > D$ are our possible orderings, so there are 3 orderings in this case. If $A > C > E > D$, then $B > A$, $A > B > C$, and $C > B > E$ are our possible orderings, so there are 3 possible orderings in this case. If $A > E > C > D$, we have that either $B > A$ or $A > B > E$, yielding 2 possible orderings. If $E > A > C > D$, we see that we must have $B > E$, yielding only one ordering. Thus, there are $\boxed{9}$ possible orderings.

The list of all nine orderings is:
$A > C > B > D > E$
$A > B > C > D > E$
$B > A > C > D > E$
$A > C > B > E > D$
$A > B > C > E > D$
$B > A > C > E > D$
$A > B > E > C > D$
$B > A > E > C > D$
$B > E > A > C > D$

12. Let circle A have radius R, where R is a positive integer between 1 and 10 inclusive. We also know that R has four distinct positive integer divisors. Find the sum of all possible areas of A.

    Solution: The positive integers between 1 and 10 inclusive that have four distinct positive integer divisors are 6, 8, and 10. Therefore, the possible values of the area of A are $6^2 \cdot \pi$, $8^2 \cdot \pi$, and $10^2 \cdot \pi$, that is, 36π, 64π, and 100π. Therefore, the sum of all possible areas of A is $\boxed{200\pi}$.

13. A six-sided die is rolled four times. What is the probability that two sixes are rolled consecutively?

    Solution: If only two sixes are rolled, they can be rolled consecutively in three ways (66xx, x66x, xx66) (x indicates an arbitrary roll of anything except for a 6). Each of these has probability of $(5/6)^2 \cdot (1/6)^2 = 25/1296$. If three sixes are rolled, two can be rolled consecutively in four ways (666x, 66x6, 6x66, x666), each with probability $(5/6) \cdot (1/6)^3 = 5/1296$. If four sixes are rolled, there is only one possible configuration (6666) with probability $(1/6)^4 = 1/1296$. So the total probability of two sixes being rolled consecutively is $3 \cdot (25/1296) + 4 \cdot (5/1296) + 1 \cdot (1/1296) = 96/1296 = \boxed{2/27}$.

14. When the elements in the set $\left\{ \frac{\pi}{2}, \frac{\pi}{2} \cdot \frac{\pi}{3}, \left(\frac{\pi}{3}\right)^2 \cdot \frac{\pi}{4} \right\}$ are simplified to lowest terms, what is the maximum value?

    Solution: The first thing to note is that writing our fractions in lowest terms won't change their values. Note that $\frac{\pi}{3} > 1$, so $\frac{\pi}{2} \cdot \frac{\pi}{3} > \frac{\pi}{2}$. Furthermore, since $6 > \pi$, $36\pi^2 > 6\pi^3$. Rearrange to get $\frac{\pi^2}{6} > \frac{\pi^3}{36}$, that is, $\frac{\pi}{2} \cdot \frac{\pi}{3} > \left(\frac{\pi}{3}\right)^2 \cdot \frac{\pi}{4}$, so $\frac{\pi}{3} \cdot \frac{\pi}{2} = \boxed{\frac{\pi^2}{6}}$ yields the maximum element in our set.

15. Let the number n be your answer for this problem. What is $6 - \frac{9}{n}$?

    Solution: Since n is the answer to our problem, we have that $\frac{6n-9}{n} = n$. Multiply both sides by n to get $6n - 9 = n^2$, or $(n-3)^2 = 0$. The only solution to this is 3, so the answer to our problem must be $\boxed{3}$.

16. Express $0.123\overline{45}$ as a fraction.

    Solution: Let $x = 0.123\overline{45}$. Then $1000x = 123 + 0.\overline{45}$. But $0.\overline{45} = \frac{45}{99} = \frac{5}{11}$. Thus, $x = \frac{123+\frac{5}{11}}{1000} = \frac{1358}{11000} = \boxed{\frac{679}{5500}}$.

17. There are four pandas being sent off to four zoos, with each panda going to a random zoo. What is the probability that one zoo gets all four pandas?

Solution: For each zoo, there is a $\frac{1}{4}$ chance of that panda going to that zoo. Thus, the probability that any particular zoo gets all four pandas is $\frac{1}{4^4}$. Since there are four zoos, our answer is $4 \cdot \frac{1}{4^4} = \boxed{\frac{1}{64}}$.

18. Kamran got an iPod touch and wanted to measure its battery life. After five runs, he recorded the following times in hours: 22.7, 19.125, 14.3, 23, and 20.875. What was the average battery life in hours?

    Solution:

$$\begin{aligned}\frac{22.7 + 19.125 + 14.3 + 23 + 20.875}{5} &= \frac{(22.7 + 14.3) + (20.875 + 19.125) + 23)}{5} \\ &= \frac{37 + 40 + 23}{5} = \frac{100}{5} = \boxed{20}\end{aligned}$$

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19. Matt randomly picks two numbers (not necessarily distinct) out of the set $\{\pi, 2, \sqrt{2}, 3.5\}$. Let p be the probability that the sum of the two numbers is an integer, and let q be the probability that the product of the two numbers is an integer. Find $p + q$.

    Solution: The numbers are not distinct; therefore, you can choose a number more than once: there are 16 combinations altogether. To solve for p, there are only two pairs that can be added together to form an integer: (2, 2) and (3.5, 3.5). Therefore p must be $\frac{2}{16} = \frac{1}{8}$.

    There are 4 pairs that yield an integer product: (2, 2), (2, 3.5), (3.5, 2), and ($\sqrt{2}$, $\sqrt{2}$). Therefore q must be $\frac{4}{16} = \frac{2}{8}$. Finally, we have $p + q = \frac{1}{8} + \frac{2}{8} = \boxed{\frac{3}{8}}$.

20. If one side of a triangle is 50 cm long and the another side is 60 cm long, how many possible integer side lengths are there for the last side?

    Solution: We have a triangle with side lengths 50, 60, and n if and only if $n < 60 + 50$, $60 < n + 50$, and $50 < n+60$ (though the last condition is redundant.) Thus, we seek the number of integer solutions to $10 < n < 110$, that is, the number of integer solutions to $11 \leq n < 110$. Every solution in that interval corresponds to some solution in the interval $[1, 100)$ (just subtract 10 from every solution in that interval), and there are 99 integers in $[1, 100)$, so our answer is $\boxed{99}$.

21. Suppose $A^2 + B^2 + C^2 = 4$ and $AB + BC + CA = 3$. Find $(A + B + C)^2$.

    Solution: $(A + B + C)^2 = A^2 + B^2 + C^2 + 2AB + 2BC + 2CA = 4 + 6 = \boxed{10}$

22. Mr. Holbrook runs around the track. He ran the first lap at 30 yards per minute. How fast must he run the next lap to increase his average speed to 45 yards per minute?

    Solution: The average speed is equal by definition to the ratio of total distance to total time. Let d be the length of the track, and let x be the speed Mr. Holbrook runs the second lap. Then the total distance is $2d$ and the total time is $\dfrac{d}{30} + \dfrac{d}{x}$. We must now solve the equation $45 = \dfrac{2d}{\frac{d}{30} + \frac{d}{x}} = \dfrac{2}{\frac{1}{30} + \frac{1}{x}} = \dfrac{60x}{x + 30}$, or $45x + 1350 = 60x$, so $x = \dfrac{1350}{15} = \boxed{90}$.

23. What is the equation of the line formed when the line $x + 2y = 5$ is reflected across the x-axis and then the y-axis? Write your answer in the form $Ax + By = C$, where A, B, and C are integers and A is positive.

    Solution: If we pick an arbitrary point $(x,\ y)$ and reflect it across the x-axis, this is equivalent to replacing y by $-y$. Similarly, reflection across the y-axis is equivalent to replacing x by $-x$. Thus, since reflecting a line is equivalent to reflecting all its points, we simply replace x by $-x$ and y by $-y$ in our equation to obtain the line $\boxed{x + 2y = -5}$.

24. Ashley likes to shop and is about to buy a pair of pants when she realizes that she is 12 dollars short of the price. She asks her friend Janet, who started out with .25 times the amount of money she did, to lend her 12 dollars. Janet, who has not spent any money, agrees and is left with 20 dollars. How much money did the two have in the beginning of their shopping trip combined?

    Solution: After Janet lent her 12 dollars, she is left with 20 dollars. Thus, she started out with 32 dollars. Since Janet started out with one fourth the amount Ashley did, Ashley must have started out with 128 dollars. Thus, in the start, they had a total of $128 + 32 = \boxed{160}$ dollars combined.

25. How many different ways are there to arrange the characters of the word "Puffball?"

    Solution 1: There are $\binom{8}{2}$ spots to place the "f"'s. There are than $\binom{6}{2}$ remaining spots to place the "l"'s. There are 4! ways to arrange the remaining letters, which have no repetition, so there are a total of $\binom{8}{2} \cdot \binom{6}{2} \cdot 24 = \boxed{10080}$ ways to arrange the characters of the word "Puffball."

    Solution 2: There is a well-known formula that gives the answer. There are 2 pairs of 2 repetitions, and there are no other repetitions, so there are $\frac{8!}{2!2!} = \boxed{10080}$ ways to arrange the characters of the word "Puffball."

26. Let $A = (3, 4)$ and $B = (11, 10)$. Rotate B 30^o counterclockwise around A to get C. What is the area of $\triangle ABC$?

    Solution: Since point B is rotated around point A to get point C, $\overline{AB}$ and $\overline{AC}$ are two radii of the circle centered at A and passing through B. Therefore, $\overline{AB} = \overline{AC}$. To calculate $\overline{AB}$, we can use the distance formula: $\overline{AB} = \sqrt{(11-3)^2 + (10-4)^2} = \sqrt{64+36} = 10$. The area of a triangle is $\frac{1}{2} \cdot b \cdot c \cdot \sin A$, where A is the angle between sides b and c. $m\angle A = 30°$, so $[ABC] = \frac{1}{2} \cdot 10 \cdot 10 \cdot \sin 30° = \frac{1}{2} \cdot 10 \cdot 10 \cdot \frac{1}{2} = \boxed{25}$.

27. If a mile is 5280 feet, what is the area, in square feet, of a 0.0625 square miles?

    Solution: If one mile is 5280 feet, then one square mile is 5280^2 square feet. Therefore, the number of square feet in 0.0625 mile^2 is just $\frac{5280^2}{16} = \left(\frac{5280}{4}\right)^2 = (1320)^2 = 1742400$, so our square has $\boxed{1742400}$ square feet.

28. Mr. Pinyan has two red balls, three blue balls, and four white balls. All balls of the same color are indistinguishable. How many ways are there of arranging the nine balls in a circle?

    Solution: Let us first consider how many ways the balls can be arranged in a line. We will then divide this number by 9 since each circular arrangement can be "unfolded" into 9 different linear arrangements, implying that there are 9 times as many linear arrangements.

    Suppose we made each of the balls distinguishable (say, for example, we call the red balls R1 and R2, the blue balls B1, B2, and B3, and the four white balls W1, W2, W3, and W4.) Then there would clearly be 9! ways to arrange our balls. Since our balls are indistinguishable, we have overcounted $2! \cdot 3! \cdot 4!$ times too many. Therefore, the number of ways to arrange our balls in a line is $\frac{9!}{4!3!2!} = 1260$. Thus, our answer is $\frac{1260}{9} = \boxed{140}$.

29. If the perimeter of a rectangle is 28 and the length of its diagonal is 10, find its area.

    Solution: Let l and w be the side lengths of the rectangle. Since the perimeter of the rectangle is 28, we have that $2l + 2w = 28$, that is, $l + w = 14$. Since the length of its diagonal is 10, by the Pythagorean theorem, $l^2 + w^2 = 100$. Adding $2lw$ to both sides, we have that $196 = (l+w)^2 = l^2 + w^2 + 2lw = 100 + 2lw$. Solving for lw, we get $\boxed{48}$.

30. Suppose $P(x) = x(x+1)(x+2)(x+3)(x+4)(x+5)(x+6)$. Find the greatest common factor of $P(1)$, $P(2)$, $P(3)$, ...

Solution: Let $d = \gcd(P(1), P(2), \ldots)$. Since $P(1) = 7! = 5040$, we get that $d \leq 5040$. However, $P(n) = 7!\binom{n}{7}$. Since $\binom{n}{7}$ is always an integer, we get that $\gcd(P(1), P(2), \ldots) \geq 7! = 5040$, so $d \geq 5040$. Therefore, $d = 5040$, so our answer is $\boxed{5040}$.

31. Find the integer x that satisfies $x^3 - 3x^2 + 3x + 124 = 0$.

    Solution: Observe that the first few coefficients of our polynomial greatly resemble the coefficients of the binomial expansion of $(x-1)^3$. Indeed, $(x-1)^3 = x^3 - 3x^2 + 3x - 1$. Thus, $x^3 - 3x^2 + 3x + 124 = 0$ may be written as $(x-1)^3 = -125$. Thus, since x is real, we have $x - 1 = -5$, so $x = \boxed{-4}$.

32. When evaluated, the difference $2^{10} - 2^9$ can be expressed in the form a^b, where a and b are integers and a is a prime number. If i is the square root of -1, what is i^{-b^a}?

    Solution: $2^{10} - 2^9 = 2(2^9) - 2^9 = 2^9 + 2^9 - 2^9 = 2^9$. Thus, $a = 2$ and $b = 9$. Hence, $i^{-b^a} = i^{-9^2} = i^{-81}$. Since $i^4 = 1$, we have $i^{-81} = i^{-81} \cdot (i^4)^{21} = i^{-81+84} = i^3 = \boxed{-i}$.

33. Find the number of positive integer solutions to the equation $2x + 3y = 101$.

    Solution: If $3y = 101 - 2x$, we see that $3|101 - 2x \rightarrow 3|2x - 101 \rightarrow 3|2x - 101 - 99 \rightarrow 3|2x - 2 \rightarrow 3|x - 1$. Furthermore, if $3|x - 1$, we have that $y = \frac{101-2x}{3} = 33 - 2\left(\frac{x-1}{3}\right)$ is an integer. Thus, a necessary and sufficient condition for x to be a solution to this equation is $3|x - 1$. Furthermore, $2x < 101$, so $x \leq 50$. Therefore, $x = 1, x = 4, \ldots, x = 46, x = 49$ give us our solutions. There are therefore $\boxed{17}$ solutions.

34. In Zhuland, quarters have radius 5 cm and dimes have radius 1 cm. If I drop a dime onto a quarter such that its center lies in the interior of the quarter (with every point in the interior having equal probability of being chosen), find the probability that the dime lies entirely within the quarter.

    Solution: The dime lies entirely within the quarter if and only if the center of the dime lies within a circle of radius of 4 concentric with the quarter. The area of this region is 16π, and the area of our whole quarter is 25π, so our answer is $\boxed{\frac{16}{25}}$.

35. If $f(x) = x^3 + ax^2 + bx + c$, $f(1) = 34$, and $f(4) = 100$, find $f(3) - f(2)$.

    Solution: $f(4) = 100$ implies that $64 + 16a + 4b + c = 100$, and $f(1) = 34$ implies that $1 + a + b + c = 34$. In other words, $16a + 4b + c = 36$, and $a + b + c = 33$. Subtracting the second equation from the first gives us $15a + 3b = 3$, or $5a + b = 1$. $f(3) - f(2) = (27 + 9a + 3b + c) - (8 + 4a + 2b + c) = 19 + 5a + b$. But $5a + b = 1$, so our final answer is $\boxed{20}$.

36. While reading a 123-paged book, Amanda decided to count the page numbers. How many times did the digit 3 show up in the page numbers?

    Solution: For the numbers from 1 through 99, 3 shows up in the units place 10 times and in the tens place 10 times. For the numbers from 101 through 123, 3 shows up 3 times in the units place, and 0 times in the tens and hundreds places, giving a total answer of $20 + 3 = \boxed{23}$.

37. Consider the unit sphere (a sphere with radius 1) centered at the origin. Find the shortest possible distance between a point on the unit sphere and the point (3, 4, 5).

    Solution: Let O be the origin, let P be the point $(3, 4, 5)$, and let Q be an arbitrary point on the surface of the sphere. We wish to minimize QP.

    By the triangle inequality, $OQ + QP \geq OP$. Since Q lies on the surface of the sphere, $OQ = 1$. Also, by the distance formula, $OQ = \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2}$. It follows that $OQ \geq 5\sqrt{2} - 1$. The distaince $5\sqrt{2} - 1$ is attainable when O, Q, and P are collinear, i.e., when Q is the intersection point of the sphere and line OP. Thus, our minimum possible distance is $\boxed{5\sqrt{2} - 1}$.

38. If a circle intersects the x-axis at points $(6,0)$, and $(8,0)$, and goes through the point $(3,2)$, find the radius of the circle.

Solution: Let our circle have equation $(x-h)^2+(y-k)^2=r^2$. We are given that the points $(6,0)$, $(8,0)$, and $(3,2)$ lie on our circle. Substituting the first two coordinates into our equation, we see that $(6-h)^2+k^2=r^2$ and $(8-h)^2+k^2=r^2$. Subtracting these two equations, we get that $(8-h)^2=(6-h)^2$. Since $8-h\neq 6-h$, $8-h=h-6$, so $h=7$. Our first equation therefore reduces to $k^2+1=r^2$. Substituting $(3,2)$ into our equation yields $(3-7)^2+(2-k)^2=r^2$, or $16+4-4k+k^2=r^2$. Since $r^2=k^2+1$, we see that $k^2+1=k^2-4k+20$, so $4k=19$, so $k=\frac{19}{4}$. Thus, $r=\sqrt{1+k^2}=\sqrt{1+\frac{361}{16}}=\sqrt{\frac{377}{16}}=\boxed{\frac{\sqrt{377}}{4}}$.

39. Michael, Alex, and Kelvin are writing a fantasy book for JP. Working at a constant rate, they can write the entire book in 196 days, 98 days, and 49 days, respectively. How many days will it take for them to finish writing the book if they work together?

Solution: Michael, Alex, and Kelvin can write $\frac{1}{196}$, $\frac{1}{98}$, and $\frac{1}{49}$ of the book, respectively, in one day. Therefore, in any day, $\frac{1}{196}+\frac{1}{98}+\frac{1}{49}=\frac{1}{28}$ of the book is completed. It will therefore take $\boxed{28}$ days for the book to be finished.

40. At which point(s) do the parabola $y=x^2-2x+1$ and the line $2x+y=1$ intersect?

Solution: We wish to solve the simultaneous system of equations $y=x^2-2x+1$ and $2x+y=1$. From our second equation, we get that $y=1-2x$. Substituting this into our first equation, we see that $-2x+1=x^2-2x+1$, which implies that $x=0$ and $y=1$. Since $(0,1)$ does lie on both the line and the parabola, our answer is $\boxed{(0,1)}$.

41. Let $f(n)$ be the remainder when n^3 is divided by 7. What is $f(f(f(3)))$?

Solution 1: There are three f's, so we simply wish to compute the remainder obtained when $3^{3^{3^3}}$ is divided by 7 (since we're just cubing 3 three times.) In other words, we wish to find $3^{3^{27}} \bmod 7$. Now, $3^{3^{27}}\equiv(3^3)^{3^{26}}\equiv 27^{3^{26}}\equiv(-1)^{3^6} \pmod 7$. But 3^6 is odd, so $(-1)^{3^6}\equiv -1\equiv 6 \pmod 7$, so our answer is $\boxed{6}$.

Solution 2: $f(3)$ is the remainder when $3^3=27$ is divided by 7, i.e. 6. $f(6)$ is the remainder when $6^3=216$ is divided by 7, i.e. 6. Therefore, $f(f(f(3)))=f(f(6))=f(6)=\boxed{6}$.

42. A blindfolded Brian is trying to pin the tail on the donkey. He knows that the donkey, a circle with radius 2ft, is on a 5ft x 5ft poster board. It is guaranteed that tail will land on the poster board. Let p be the probability that he pins the tail on the donkey and q be the probability that he misses the donkey. What is $p+q$?

Solution: The probability that Brian pins the tail on the donkey and the probability that he doesn't should add up to 100%. He can only put the pin on and the board and the board consists of areas that consist of a hit and of a miss. Any probability should add up to $\boxed{1}$ (100%) assuming no external actions are taken to change the course of events.

43. How many integers from 1 to 1000 are multiples of 12 but not 15?

Solution: Suppose we have that $12|n$ but $15 \not| n$. The first condition tells us that $3|n$ and $4|n$. The second tells us that $3 \not| n$ or $5 \not| n$. Since $3|n$, we see that $5 \not| n$. Thus, $12|n$ and $15 \not| n$ iff $3|n$, $4|n$, and $5 \not| n$.

We will count the number of multiples of 12 between 1 and 1000 inclusive, and then subtract from that count the number of multiples of 12 between 1 and 1000 that are also multiples of 5, i.e., the number of multiples of 60 between 1 and 1000 inclusive. However, there are just $\lfloor\frac{1000}{12}\rfloor=83$ multiples of 12 between 1 and 1000 inclusive, and there are just $\lfloor\frac{1000}{60}\rfloor=16$ multiples of 60 between 1 and 1000, inclusive. Thus, our answer is just $83 - 16 = \boxed{67}$ integers from 1 to 1000 that are multiples of 12, but not of 15.

44. Paul built a very mathematical snowman. The snowman consisted of three spheres whose radii formed an increasing geometric sequence with common ratio $\frac{3}{2}$. If the volume of the top sphere was 171π less than that of the bottom sphere and the radius of the smallest sphere can be written as $a\sqrt[3]{\frac{b}{c}}$ (where a, b, and c are positive integers, $\frac{b}{c}$ is in lowest terms, and b and c do not contain any perfect cube factors other than 1), what is $a + b + c$?

    Solution: Let the top sphere have radius x. Then the middle sphere has radius $\frac{3}{2}x$ and the largest sphere has radius $\frac{9}{4}x$. The volume of a sphere is $\frac{4}{3}\pi r^3$. Therefore, the smallest sphere has a volume of $\frac{4}{3}\pi x^3$, and the largest has a volume of $\frac{243}{16}\pi x^3$. Then we have $\frac{4}{3}\pi x^3 + 171\pi = \frac{243}{16}\pi x^3$. We can cancel π and subtract $\frac{4}{3}x^3$ from both sides to get $171 = \frac{243}{16}x^3 - \frac{4}{3}x^3 = \frac{665}{48}x^3$, and solving for x^3 we get $x^3 = \frac{171 \cdot 48}{665} = \frac{432}{35}$. This implies that $x = 6\sqrt[3]{\frac{2}{35}}$, so $a = 6$, $b = 2$, and $c = 35$. Then $a + b + c = 6 + 2 + 35 = \boxed{43}$.

45. A 68-digit number β, when written in binary (base 2), is made up of sixty-eight 1's. Find $\log_2(1 + \beta)$.

    Solution: $\beta = \underbrace{111\ldots11}_{68}{}_2 = 2^0 + 2^1 + \ldots + 2^{67} = 2^{68} - 1$. Thus, $\log_2(1 + \beta) = \log_2(2^{68}) = \boxed{68}$.

46. How many factors does 84 have?

    Solution 1: $84 = 2^2 \cdot 3^1 \cdot 7^1$, so our answer is $(2 + 1)(1 + 1)(1 + 1) = \boxed{12}$.

    Solution 2: We can just list out the factors. They are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84, so there are $\boxed{12}$ different factors.

47. Michael and Jordan are competing in a basketball duel. Mike has a $\frac{2}{3}$ chance of winning a match and Jordan has a $\frac{1}{3}$ chance of winning a match (no ties are allowed). What is the probability that Jordan will win in three or fewer matches if they play five matches?

    Solution: There is exactly one way Jordan can win all five matches (JJJJJ), and this has a probability of $\left(\frac{1}{3}\right)^5 = \frac{1}{243}$. There are five ways Jordan can win four matches (JJJJM, JJJMJ, JJMJJ, JMJJJ, MJJJJ), each having a probability of $\frac{2}{3}\left(\frac{1}{3}\right)^4 = \frac{2}{243}$. Therefore, the probability that Jordan will win four matches is $5 \cdot \frac{2}{243} = \frac{10}{243}$, so the probability that Jordan wins at least four games is $\frac{1}{243} + \frac{10}{243} = \frac{11}{243}$. The probability that the opposite will happen (i.e. that Jordan will win three or less games) is therefore $1 - \frac{11}{243} = \boxed{\frac{232}{243}}$.

48. Evaluate $1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{1 + \cdots}}}$.

    Solution: Let $x = 1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{1 + \cdots}}}$. Since x is also the denominator of the fraction, we substitute x for that, resulting in $x = 1 + \frac{2}{x}$. Multiply both sides by x to clear the denominator, resulting in

$x^2 = x + 2$. We see that -1 and 2 are the solutions to the quadratic equation, but $x \neq -1$ because the original expression only contained addition of positive numbers. Therefore, $x = \boxed{2}$.

49. The numbers 1-4 are written on a board. At any time, we may pick three numbers a, b, and c on the board and replace them with $\frac{a+b}{2}$, $\frac{b+c}{2}$, and $\frac{c+a}{2}$. If M and m are the greatest and smallest sums of the four numbers that can be attained after doing the operation twice, respectively, then find $m \cdot M$.

Solution: Observe that the sum of the numbers on the board stays the same after such an operation is performed. Thus, M and m are both equal to 10, so our answer is $\boxed{100}$.

50. Find the minimal value of

$$\sqrt{(a-1)^2 + (b-1)^2} + \sqrt{(a-1)^2 + b^2} + \sqrt{a^2 + (b-1)^2} + \sqrt{a^2 + b^2}$$

where a, b are arbitrary real numbers.

Solution 1: Let $A = (a-1, b-1)$, $B = (a-1, b)$, $C = (a, b-1)$, $D = (a, b)$, $X = (a - \frac{1}{2}, b - \frac{1}{2})$, and $O = (0, 0)$. First, observe that A, B, C, and D are the vertices of a square with center X. Second, observe that our sum is equal to the sum of the distances $OA + OB + OC + OD$. By the triangle inequality, $OA \geq OX + AX$, $OB \geq OX + BX$, $OC \geq OX + CX$, and $OD \geq OX + DX$. But $AX = BX = CX = DX$, as they are all half the lengths of the diagonal of the square. Since the square has side length one, we have that $AX + BX + CX + DX = 2\sqrt{2}$. Adding up the inequalities we have, we see that $OA + OB + OC + OD \geq 2\sqrt{2} + 4OX$. But setting $X = O$ yields $OX = 0$, so we see that $OA + OB + OC + OD \geq 2\sqrt{2}$. We see that this value is attained when $a = b = \frac{1}{2}$, so our minimum attainable value is $\boxed{2\sqrt{2}}$.

Solution 2: Write this as $\sqrt{(1-a)^2 + (1-b)^2} + \sqrt{(1-a)^2 + b^2} + \sqrt{a^2 + (1-b)^2} + \sqrt{a^2 + b^2}$. The triangle inequality in vector form states that $\sqrt{p^2 + q^2} + \sqrt{r^2 + s^2} \geq \sqrt{(p+r)^2 + (q+s)^2}$. Applying this inequality three times, we see that our sum is greater than or equal to $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. Setting $a = b = \frac{1}{2}$ attains this minimum, so we see that our final answer is $\boxed{2\sqrt{2}}$.