7th Grade Solutions

1. Dividing 20 by $\frac{1}{2}$ is equivalent to multiplying 20 by 2, which is 40. Adding 30 to 40, the answer is **70**.

2. $20-(20-(20-(20-(20-(20-(20))))))=20-(20-(20-(20-(20-0))))=20-(20-(20-(20-20)))=20-(20-(20-0))=20-(20-20)=20-0=$ **20**

3. Increasing the width by 10% is equivalent to multiplying the width by $\frac{11}{10}$. Similarly, reducing the length by 10% is equivalent to multiplying the length by $\frac{9}{10}$. Hence, the total area is multiplied by $\frac{11}{10}\times\frac{9}{10}=\frac{99}{100}$. Therefore, the area of the rectangle is decreased by **1%**.

4. Since $\overline{AO}$ and $\overline{BO}$ are both radii of the given circle, we have that $\overline{AO}=\overline{BO}$, which implies $x+3=3x$. This gives $x=\mathbf{\frac{3}{2}}$.

5. We have a direct proportion with the number of frogs and the number of flies, and we have an indirect proportion with the number of frogs and the time it takes. To verify this, consider the cases of 6 frogs and 3 flies, and 6 frogs and 3 minutes. Therefore, the combined proportion is $k=\frac{GM}{L}$ where G is the number of frogs, L is the number of flies, and M is the number of minutes. We insert the case $G=3$, $L=3$, $M=3$ and we get that $k=3$. Inserting the case $G=6$, $L=6$, we see that $M=\mathbf{3}$.

6. Prime factorizing 2010, we have $2\times 3\times 5\times 67$, so there are 4 distinct factors. For each of these prime factors, we have the option of using them or not (2 options) to compose factors of 2010. Therefore, the total number of ways by the multiplication principle, is $2\times 2\times 2\times 2=\mathbf{16}$.

7. Writing the equations in slope-intercept form, we find that the equations of the two lines are $y-28=7(x-10)$ and $y-28=4(x-10)$. The x-intercepts are the x-values that occur when $y=0$, so the x-intercept of the first line is $\frac{-28}{7}+10=6$ and the x-intercept of the second is $\frac{-28}{4}+10=3$, so our answer is $6-3=\mathbf{16}$.

8. $\sqrt{1000}=\sqrt{100\cdot 10}=\sqrt{100}\cdot\sqrt{10}=10\sqrt{10}$. No perfect square (aside from 1) divides 10, so our answer is **10**.

9. The divisibility rule for 3 states that the sum of the digits must be divisible by 3. The sum of the digits of the number $12A4$ is $1+2+A+4=7+A$. The only digits that will make $7+A$ divisible by 3 are 2, 5, and 8. $2+5+8=\mathbf{15}$.

10. The greatest common factor between 24 and 21 is 3, which can be seen if the two number are factored. The rule for least common multiple between two number a and b is $a\times b\div GCF$, where GCF is the greatest common factor between a and b. This yields the answer **168**.

11. The numbers 1 through 9 all use one digit. The numbers 10 through 20 all use two digits. There are 9 numbers from 1 through 9, and there are 11 numbers from 10 through 20. $9\times 1+11\times 2=\mathbf{31}$.

12. If the 3 action figures must go first must find the ways she can throw out just the action figures. There are 3 action figures, so the total number of ways to order them to be thrown out is 3! or $3\times 2\times 1=6$. There are 6 remaining objects, all of which are distinct, so the total number of ways to order them to be thrown out is 6! or $6\times 5\times 4\times 3\times 2\times 1=720$. The total number of ways is $6\times 720=\mathbf{4320}$.

13. The number 1337 is to be multiplied to itself many times. Since we are only concerned with the last digit, we will only concern ourselves with the last digit of 1337, which is 7 (the other digits cannot affect the ones digit in any way). 7, when raised to different powers, follows a pattern, which can easily be verified (keep in mind that only the ones digit is of any concern). 7^1 has a ones digit of 7, 7^2 has a ones digit of 9, 7^3 has ones digit of 3, 7^4 has a ones digit of 1, and 7^5 has a ones digit of 7. This cycle repeats with a period of 4. We are to raise 1337 to the 101st power, which means that it will complete the cycle in the ones digit 25 times ($101\div 4=25R1$). However, there is a 1 remainder, which means that it will eventually end up on the first number in the sequence, which is **7**.

14. Adding the two equalities, we obtain $3x+3y+3z=9$. Dividing both sides of it by 3, we have $x+y+z=\mathbf{3}$.

15. Consider an equilateral triangle ABC with median $\overline{AD}$. Because of the symmetry in the drawing, angle $ABD = 60°$ and angle $BAD = 30°$. Then triangle ABD is a 30-60-90 triangle, and the length of $\overline{BD}$ is one-half that of $\overline{AB}$, or **13**.

16. If both trains are traveling towards each other at 40 mph, then the rate at which the distance between the two of them is closing is $40 + 40 = 80\ mph$. The total time it takes for the buses to meet is $\frac{160\ miles}{80\ mph}$, which is 2 hours. Since the butterfly will fly for the entire duration, it will fly a total distance of $45\ mph \times 2\ hours$, or **90** miles.

17. Add the times: 8 hours and 52 minutes (starting time) + 4 hours and 37 minutes (time to get to the moon) + 2 hours and 42 minutes (time stayed at the moon) + 8 hours 74 minutes (time to get back to the Earth) = 25 hours and 25 minutes from 12:00 A.M. or **1:25 A.M.**

18. Each hand shake requires 2 people, so the total number of handshakes would be the total number of distinct ways to choose 2 people from a group of 10. Using the choose function, ${}_{10}C_2 = 45$. Without using the choose function, consider that to choose 2 people from a group of 10, there are 10 choices for the first person, and 9 choices for the second, since the first person cannot shake hands with himself. Thus, our initial answer is 90. However, given the fact that choosing, for example, Bob first and then Dave, is the same as choosing Dave and then Bob, we realize that we have actually counted each handshake twice. Thus, we divide by 2, and the answer is **45**.

19. Consider how much of the lawn is mowed in just one hour working together. Albert can mow a lawn in 6 hours, and so in 1 hour he will have moved $\frac{1}{6}$ of one lawn. Alex, by the same logic, will have moved $\frac{1}{4}$ of one lawn. Thus their combined effort for one hour is $\frac{1}{6}+\frac{1}{4} = \frac{5}{12}$. If one hour will yield $\frac{5}{12}$ of a lawn, then $\mathbf{\frac{12}{5}}$ of an hour will yield one whole lawn ($rate \times time = result$).

20. First find the total number of ways to arrange everyone in a line, including ways when David is in front or at the end. This is 5!, or $5 \times 4 \times 3 \times 2 \times 1 = 120$, since there are 5 choices for the first position, 4 choices for the second position, etc. The total number of ways to arrange everyone in a line when David is at the front is 4!, since David has claimed one spot, and the others must fill into the 4 spots; $4! = 24$. This is the same as the number of ways to arrange everyone in a line when David is at the end; 24. Thus we take our total number, 120, and subtract 24 twice: $120 - 24 - 24 =$ **72**.

21. The formula to count the interior angles of the polygon with n sides is $180(n - 2)$. This can be verified by drawing a point in the middle of any polygon and forming triangles by drawing a line from every vertex to the center point. The total number of triangles formed is equal to the number of sides. Triangles have a total angle sum of 180, so the total number of degrees is $180n$. However, since 360 degrees is in the center of the polygon where all the triangles meet, we subtract 360: $180n - 360$. Thus by distribution property, we get $180(n - 2)$. Since an octagon has 8 sides, we must multiply 6 by 180 to get **1080**.

22. The word "puppy" has 5 letters. The total way to arrange 5 objects is 5! or $5 \times 4 \times 3 \times 2 \times 1 = 120$. However, we must account for the duplicate cases that arise from the fact that the letter "p" is indistinguishable from other letter p's. For every arrangement, there are 3! versions (total number of ways to arrange the 3 p's). Thus, we take only one out of every $3! = 6$ of these by dividing 120 by 6. The final answer is **20**.

23. We factor out the polynomial to get $(x - 4)(x - 5)$. Therefore, the polynomial will have factors $(x - 4)$ and $(x - 5)$ for any value of x. Since the polynomial must be prime for valid values of x, either $(x - 4)$ or $(x - 5)$ must be 1 (or -1), so that the value can remain prime. If we let $x - 4 = 1$, then $x = 5$, and we find that the product is 0 ($x - 5 = 0$ for $x = 5$). Thus we set $x - 5 = 1$, then $x = 6$. The product is $(x - 4)(x - 5)$, or $(2)(1) = 2$, which is prime. By the same logic, for -1, we find this case works only when $x - 4 = -1$. We then have **2** values of x for which the expression is prime. *O*R We factor out the polynomial to get $(x - 4)(x - 5)$. From this, we know the polynomial must be even, since either $(x - 4)$ or $(x - 5)$ must be even and the other must be odd. The only even prime is 2, so the factors $(x-4)$ and $(x-5)$ must be satisfied by $(2, 1)$ or $(-1, -2)$, respectively. $(1, 2)$ and $(-2, -1)$ have no solutions. There are **2** values of x for which the expression is prime.

24. Let us take the worst case scenario, which is that Alex will always draw socks of a different color. The first sock will be different from the second sock, which will be different from the third sock, which will be different

from the fourth sock, which will be different from the fifth sock, etc. However, if we assign colors to the first four socks (white, purple, pink, and orange respectively [actual color does not matter]), we see that there is no other way to draw a sock of a different color for the fifth sock. Therefore, Alex will need to draw **5** socks to be guaranteed 2 socks of the same color.

25. The problem states that the circle was split into four equal sections. If one of the sections has an area 16π, then the total circle has area $4 \times 16\pi = 64\pi$. The formula for area of the circle is $\pi \times r^2$, where r is the radius. $\pi \times r^2 = 64\pi$; $r^2 = 64$; $r = 8$. Thus we have that the radius is equal to 8. Since the diameter of a circle is twice the length of the radius, the circle's diameter is **16**.

26. First of all, we compute the value of $g(3)$. Using the given formula for $g(x)$, we have that $g(3) = \frac{5 \cdot 3+12}{3^3} = \frac{27}{27}$ $= 1$. It remains to find the value of $f(1)$, which is just $9 \times 1^2 + 5 = \mathbf{14}$.

27. The angle between the hour hand and 6 O'clock is $180 - (30 + \frac{30}{60} \times 37) = 131.5°$. The angle between 6 O'clock and the minute hand is $6 \times 7 = 42°$. $131.5 + 42 = \mathbf{173.5}°$

28. Angle between minute hand and 12 o'clock: $180 - 42 = 138°$ Angle between 12 o'clock and hour hand: $30 + 0.5 \times 37 = 48.5°$. Therefore, the minute hand has to catch up $138 + 48.5 = 186.5°$. Note that the hour hand moves 0.5○ a minute and the minute hand moves 6 degrees a minute. Hence, the minute hand catches up the hour hand $5.5°$ a minute. $\frac{186.5}{5.5} = \mathbf{\frac{373}{11}}$.

29. In any quadrilateral with A, B, and C as vertices, either $\overline{AB}, \overline{BC}$, or $\overline{CA}$ must be a diagonal. If $\overline{AB}$ is a diagonal, reflect C across the midpoint of $\overline{AB}$ to obtain the fourth point. We can construct two other points in a similar fashion. Therefore, the answer is **3**.

30. By the Pythagorean theorem, side $\overline{AC}$ has a length of 10. Let x be the length of $\overline{BD}$. The area of the triangle can be obtained by the formula $\frac{1}{2} \times b \times h = \frac{1}{2} \times 6 \times 8 = 24$. Also, the area can be obtained by $\frac{1}{2} \times 10 \times x$. Thus, if these two are equated, $x \times 10 \times \frac{1}{2} = 24$. Solve for x, and it equals **4.8**.

31. Note that in every game, one participant is eliminated. In order for one participant to remain, there must be **2009** games.

32. The last digts of 1!, 2!, 3!, and 4! are 1, 2, 6, and 4 respectively. Once we reach 5!, the products have at least one factor 5 and one factor 2, which means they are multiples of 10, and thus have 0 in the ones digit. Thus we only concern ourselves with the first 4, which has a sum of 13, or **3** in the ones digit.

33. The mode is obviously 9. So the problem comes down to finding x such that the mean and median sum to 14. The sum of the numbers is $42 + x$, so the mean is $6 + \frac{x}{7}$. Hence, the median is $14 - (6 + \frac{x}{7})$, so the median equals $8 - \frac{x}{7}$. Since $x > 0$ the median cannot be 9, so the median is either 6 or x. If 6 is the median then the mean is less than 6 so the sum of the mean and median is less than 12. Therefore, the median is x. Solving $8 - \frac{x}{7} = x$ gives $x = \mathbf{7}$.

34. The area Weili can travel can be divided into two sections. The first is a semicircle with its center at Point A and a radius of 5. The second is the area Weili can travel after it goes around the intersection of Wall B and Wall C, which is a quarter of a circle with radius 2. Using the equation for the area of the circle, we get $5^2 \times \frac{\pi}{2} + 2^2 \times \frac{\pi}{4} = \mathbf{\frac{27\pi}{2}}$.

35. Since $x : y : z = 2 : 3 : 5$, there exists a number a such that $x = 2a, y = 3a$, and $z = 5a$. Plugging this into the condition $x^2 + y^2 + z^2 = xyz$, we have $4a^2 + 9a^2 + 25a^2 = 30a^3$. As x, y, z are nonzero, we also have that a is nonzero. Hence, we may divide both sides of the equation by $2a^2$, which gives $19 = 15a$, so $a = \frac{19}{15}$. Hence, $x + y + z = 2a + 3a + 5a = 10a = \mathbf{\frac{38}{3}}$.

36. Let the side length of the given cube be x. Obviously, the volume of the cube is equal to x^3. Since the cube has six faces and each of those faces' area is equal to x^2, its surface area is $6x^2$. Hence, we have $x^3 = 6x^2$, and solving this equation we get $x = \mathbf{6}$.

37. Without loss of generality, let us assume that the distance between his home and his school is 12 miles. Then, it took him two hours on his way to school and three hours his way back from school. Hence, he traveled the distance of 24 miles in 5 hours. Therefore, his average speed on his round trip is **4.8** miles/hour.

38. The distance traveled is just the length of the train, which is 50 meters. Since the train is traveling at 36 kilometers per hour, their combined rate would mean that they move 72 kilometers in reference to each other: (72000 meters per hour) $\times$ (time he took) = 50 meters, which means that the time is $\frac{1}{1440}$ hours = **2.5** seconds.

39. In total, Dan walked 10 miles in 80 minutes. This is equal to 10 miles in $\frac{4}{3}$ hours, and this implies that Dan's average speed for the whole trip is equal to $\mathbf{\frac{15}{2}}$ miles per hour.

40. Let x be the amount of grass originally in the field. Let y be the amount of grass grown in one day. Then, we have $x + 14y = 9 \times 14$ and $x + 10y = 11 \times 10$. Solving this system of equations, we find $y = 4$ and $x = 70$. If n is the number of days the cows in the third field take to eat all the grass, $70 + 4 \times n = 14 \times n$, which yields $n = \mathbf{7}$.

41. Let the radius of this circle be r. Then, its area and its circumference is equal to πr^2 and $2\pi r$, respectively. From the given condition, we obtain $\pi r^2 : 2\pi r = 5 : 1$, which gives $r = \mathbf{10}$.

42. Notice that the sum of the first n positive odd integers is equal to n^2. Since $1 + 3 + 5 + \times + 4001$ is the sum of first 2001 positive odd integers, its sum is equal to 2001^2. Hence, $\sqrt{1 + 3 + 5 + \cdots + 4001} = \mathbf{2001}$.

43. The given expression is equal to $(13 \times 11 \times 7)^{2010} \times 11 \times 7^2 = 1001^{2010} \times 11 \times 7^2$. So we just need to find the units digit of 11×7^2, which is **9**.

44. Drawing a clock makes this problem much easier. After putting the hour hand at 22 minutes, it is easy to see that the hour is 4. The hour hand is 2 minutes past 4, telling us that the hours is $\frac{2}{5}$ done (there are 5 intervals between 4 and 5, these intervals usually being the minutes). By taking $\frac{2}{5}$ of an hour (60 minutes), we find that the time is 24 minutes after the hour, so our answer is **4:24**.

45. If you add one to the number that we are trying to find, it will be divisible by 2, 3, 4, and 5. The lowest possible number to do that is 60. Since we added one, now we must subtract one. $60 - 1 = \mathbf{59}$.

46. A three-digit palindrome is determined by its first two digits (since the third digit must be equal to the first). There are 9 choices for the first digit (1 through 9) and 10 choices for the second digit (0 through 9). Thus, there are $9 \times 10 = \mathbf{90}$ three-digit palindromes.

47. $2 \times 3^4 + 0 \times 3^3 + 1 \times 3^2 + 1 \times 3^1 + 0 \times 3^0 = 162 + 9 + 3 = \mathbf{174}$

48. At the end of 11 movements, the cow must either be on vertex B or vertex D. Also, at each second, the cow has 2 choices, and thus the number of total paths is 2^{11}. However, since half of these paths will result in the cow on vertex D, our answer is $\frac{2^{11}}{2} = 2^10 = \mathbf{1024}$.

49. We have $f(x) = \frac{1}{(2x-1)(2x+1)} = \frac{1}{2}\left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)$. Hence,
$f(1) + f(2) + \times + f(2010) = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{4017} - \frac{1}{4019} + \frac{1}{4019} - \frac{1}{4021}\right) = \frac{1}{2} \times \frac{4020}{4021} = \mathbf{\frac{2010}{4021}}$.

50. Let's compute the Fibonacci numbers mod 4: it is $1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, \ldots$ Therefore,$i^{F_1} + i^{F_2} + \cdots + i^{F_{2009}} + i^{F_{2010}} = 335(i^1 + i^1 + i^2 + i^3 + i^1 + i^0)$. Computing this, we have $335(2i) = \mathbf{670i}$.