1. Recall that $10^n = \underbrace{100\ldots00}_{n \text{ zeros}}$, hence $10^{10} + 10^8 + 10^6 + 10^4 + 10^2 + 10^0 =$ $\boxed{10101010101}$.

2. In general, the number exactly halfway between a and b is $\frac{a+b}{2}$ - the average of the two numbers. Therefore, the number halfway between $\frac{1}{6}$ and $\frac{1}{4}$ is:
$$\frac{\frac{1}{6}+\frac{1}{4}}{2} = \frac{\frac{2}{12}+\frac{3}{12}}{2} = \frac{\frac{5}{12}}{2} = \boxed{\frac{5}{24}}.$$

3. Let the number in question be x. Then we know
$$21 + \frac{1}{4}x = \frac{3}{5}x \implies 21 = \left(\frac{3}{5} - \frac{1}{4}\right)x$$
$$\implies 21 = \left(\frac{12}{20} - \frac{5}{20}\right)x = \left(\frac{7}{20}\right)x$$
$$\implies x = 21 \cdot \frac{20}{7} = \boxed{60}.$$
Verifying, $\frac{1}{4}$ of 60 is 15, and $\frac{3}{5}$ of 60 is 36. As $21 + 15$ is indeed equal to 36, this check is successful.

4. Except for the middle letter, every palindrome must contain an even number of every present letter. Suppose the letter "b" is the middle letter. Then there must be at least 2 g's, at least 2 t's, at least 2 r's, and at least 2 o's. Hence there are at least $1 + 2 + 2 + 2 + 2 = \boxed{9}$ letters in such a palindrome. One example is the palindrome "rtgobogtr".

5. There are 16 words in the sentence, and exactly 5 of them have four letters, as shown: "**What** is the probability **that** a randomly chosen **word** of **this** sentence has exactly **four** letters?"

   Therefore, the desired probability is simply $\boxed{\frac{5}{16}}$.

6. Since 2015 is not a leap year, there are exactly 365 days between October 12th, 2015 and October 12th, 2014. Since 364 is a multiple of 7, after 364 days it will still be a Sunday. Hence, October 12th, 2015 will be one day after a Sunday, or $\boxed{\text{Monday}}$.

7. Applying the definition, we have $2\#0 = 2 \cdot 0 - 2 - 3 = -5$ and $1\#4 = 1 \cdot 4 - 1 - 3 = 0$. Thus,
$$(2\#0)\#(1\#4) = 5\#0 = (-5) \cdot 0 - (-5) - 3 = \boxed{2}.$$

8. Notice that $9 \cdot 18 = 162$ and $9 \cdot 19 = 171$. Any other multiple of 9 will be further away from 169, so our answer is the closer of the two - $\boxed{171}$.

9. We have $P(1) = (1+1)(1+2)(1+3)\ldots(1+2013)(1+2014) = 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 2014 \cdot 2015$ and $P(0) = (0+1)(0+2)(0+3)\ldots(0+2013)(0+2014) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2013 \cdot 2014$. Thus,

$$\frac{P(1)}{P(0)} = \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \ldots \cdot \cancel{2014} \cdot 2015}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \ldots \cdot \cancel{2013} \cdot \cancel{2014}} = \frac{2015}{1} = \boxed{2015}.$$

Alternatively, the exact same cancellation shows that $\frac{P(x)}{P(x-1)} = \frac{x+2014}{x}$, hence $\frac{P(1)}{P(0)} = \frac{1+2014}{1} = 2015$.

10. We can prime factorize $12 = 2^2 \cdot 3$ and $18 = 2 \cdot 3^2$. Hence, since the greatest common divisor takes the smallest exponent of each prime, we have $\gcd(12, 18) = 2 \cdot 3 = 6$. Similarly, since the least common multiple takes the largest exponent of each prime, we have $\text{lcm}(12, 18) = 2^2 \cdot 3^2 = 36$. Thus our answer is $6 + 36 = \boxed{42}$.

    Alternatively, we can utilize the Euclidean Algorithm to find $\gcd(12, 18) = \gcd(12, 6) = \gcd(6, 6) = 6$, then use $\text{lcm}(x, y) = \frac{xy}{\gcd(x, y)}$ to find $\text{lcm}(12, 18) = \frac{12 \cdot 18}{6} = 36$. Our answer is then the same as before.

11. Since 2 darps is equal to 4 derps, 6 darps is equal to 12 derps. Similarly, since 3 derps is equal to 5 dirps, 12 derps is equal to 20 dirps. Putting these together, 6 darps is equal to $\boxed{20}$ dirps.

12. Suppose the side length of the cube is s. Then the area of each face of the cube, being a square, has area s^2. Since there are 6 such faces, the surface area of the cube is $6s^2$. As such we have $6s^2 = 294 \implies s^2 = 49 \implies s = \pm 7$. Clearly $s = -7$ does not make sense as a side length, leaving the solution $s = \boxed{7}$.

13. Let there be n nickels and d dimes. Since there are 70 coins in total, the total value of which is 555 cents, we have the system of equations:

$$n + d = 70 \implies 5n + 5d = 350$$

$$5n + 10d = 555$$

    Subtracting the first equation from the second gives $5d = 205 \implies d = 41$, implying that $n = 29$. Checking, there are indeed 70 coins, and the nickels are worth $\$0.05 \cdot 29 = \$1.45$ and the dimes are worth $\$0.10 \cdot 41 = \$4.10$, which does indeed sum up to $5.55. Therefore our answer is $d - n = 41 - 29 = \boxed{12}$.

14. Let a and b be the integers, and without loss of generality assume that $a \geq b$ (if $a < b$, we can simply switch the two). Since $8 = a + b \leq 2a$, we know that a is at least 4. Furthermore, since $a^2 \leq a^2 + b^2 = 34$, we know that a is less than 6. Therefore we only need to check $a = 4$ and $a = 5$, the latter of which leads to the solution $a = 5, b = 3$. Therefore, $ab = 5 \cdot 3 = \boxed{15}$.

15. Cross-multiplying, the equation is equivalent to $(x^2 + 1)(x + 1) = (x^2 - 1)(x - 1)$. Expanding gives $x^3 + x^2 + x + 1 = x^3 - x^2 - x + 1 \implies 2(x^2 + x) = 0 \implies x^2 + x = 0 \implies x(x + 1) = 0$. Thus $x = 0$ or $x = -1$. However, if $x = -1$, then the fraction $\frac{x^2 - 1}{x + 1}$ will have denominator zero, meaning that the only valid solution is $x = \boxed{0}$.

16. To mix the color purple, Rita would have to roll red once and blue once. She can either roll red first and blue next, which has a probability of $\frac{3}{6} \cdot \frac{1}{6} = \frac{1}{12}$, or roll blue first and red next, which also has a probability of $\frac{1}{6} \cdot \frac{3}{6} = \frac{1}{12}$. The final probability is thus $\frac{1}{12} + \frac{1}{12} = \boxed{\frac{1}{6}}$.

17. Suppose there are a alpacas and c chickens. Considering the heads of the animals gives us the equation $a + 2c = 94$, and considering their legs gives us the equation $4a + 2c = 238$. Subtracting the first equation from the second gives us $3a = 238 - 94 = 144 \implies a = 48$, and hence $c = 23$. As such there are $a + c = 48 + 23 = \boxed{71}$ animals on the farm.

18. Notice that $4^x - 1 = (2^x)^2 - 1 = (2^x - 1)(2^x + 1)$, meaning that $4^x - 1$ can be prime only if $2^x - 1 = 1$ or $2^x + 1 = 1$. The latter is impossible as there is no positive integer x such that $2^x = 0$, while the former leads to $2^x = 2 \implies x = 1$. This is the only such value, and thus our answer is $\boxed{1}$.

    Alternatively, notice that 4^x always leaves a remainder of 1 upon division by 3, and thus $4^x - 1$ is always divisible by 3. As such, the only way $4^x - 1$ can be prime is if $4^x - 1 = 3 \implies 4^x = 4 \implies x = 1$, and the rest is as before.

19. Note that the surface area varies with the square of the side length, and the volume varies with the cube of the side length. In other words, multiplying the side length by x is the same as multiplying the surface area by x^2 and the volume by x^3. In this case, we are adding 20% to the side length, equivalent to multiplying by 1.2. Therefore, the surface area of the cube is increased by a factor of 1.44, or 44%, and the volume of the cube is increased by a factor of 1.728, or 72.8%. Thus $x = 44$ and $y = 72.8$, meaning $5(y - x) = 5(72.8 - 44) = \boxed{144}$.

20. It is not difficult to guess-and-check the solution, but we will investigate a more general method. From $ab + cd = a + b + c + d$ we get $(ab - a - b) +$

$(cd - c - d) = 0 \implies (ab - a - b + 1) + (cd - c - d + 1) = 2$. Therefore, $(a-1)(b-1) + (c-1)(d-1) = 2$. If none of a, b, c, d are 1, then we would necessarily have $(a-1)(b-1) = 1$ and $(c-1)(d-1) = 1$ as a, b, c, d are positive integers. This would give us $a = b = c = d = 2$, which does not satisfy either equation. Thus we can assume $a = 1$. This implies that $(c-1)(d-1) = 2$, implying that $c = 3$ and $d = 2$ or the reverse. Finally, $a + b + c + d = 11$ gives us $1 + b + 3 + 2 = 11 \implies b = 5$. Finally, $abcd = 1 \cdot 5 \cdot 3 \cdot 2 = \boxed{30}$.

21. There are two cases to consider. The first is when Alex wins on his first turn, and the second is when Alex wins on his second turn. Notice that Alex will never get a third turn, as by then all the spots will necessarily be winning. The first case necessitates Kelvin not rolling an even number on his first turn, which occurs with a probability of $\frac{1}{2}$, and Alex rolling either an even number or the number Kelvin rolled, which occurs with a probability of $\frac{2}{3}$. Hence the probability of this case occurring is $\frac{1}{3}$. The second case requires Kelvin rolling an odd number, followed by Alex not winning, followed by Kelvin rolling the only number remaining that isn't winning. Then all 6 numbers will be winning, and Alex will win. The probability of this case is $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{36}$, making the total probability $\frac{1}{3} + \frac{1}{36} = \boxed{\frac{13}{36}}$.

22. The biking leg of his trip is completed in $\frac{2}{10} = \frac{1}{5}$ of an hour, or 12 minutes, while the swimming leg of his trip is completed in $\frac{3}{12} = \frac{1}{4}$ of an hour, or 15 minutes. Since the entire trip takes $\frac{6}{6} = 1$ hour, he should spend $60 - 12 - 15 = \boxed{33}$ minutes on his walk.

23. Since the triangle's altitudes are all equal, the triangle must be equilateral. Suppose the side length of this triangle is $2s$. Then, by the Pythagorean theorem, $s^2 + 6^2 = (2s)^2 \implies 36 = 3s^2 \implies 12 = s^2$. The area of the triangle is thus $\frac{s^2\sqrt{3}}{4} = \frac{12\sqrt{3}}{4} = \boxed{3\sqrt{3}}$.

24. The only way that this is possible is if the children alternate in the form "GBGBGBG", where "G" denotes a girl and "B" denotes a boy. The 4 girls can arrange themselves in any of $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways, while the 3 boys can arrange themselves in $3! = 3 \cdot 2 \cdot 1 = 6$ ways. Thus there are a total of $6 \cdot 24 = \boxed{144}$ possible arrangements.

25. Let the number be $\overline{abc}$. If any of a, b, c are zero, then the product of the digits would be zero, implying that the sum of the digits is 0 - impossible. Hence a, b, c are all positive digits. WLOG assume that $a \leq b \leq c$ - we'll account for the permutations later. Thus we have $abc = a + b + c \leq 3c \implies ab \leq 3$, meaning we need check only $(a, b) = (1, 1)$, $(a, b) = (1, 2)$,

and $(a, b) = (1, 3)$. The former case gives us $c = 2 + c$, contradiction, the second gives us $2c = 3 + c \implies c = 3$, and the third gives us $3c = 4 + c \implies c = 2$, a contradiction as we assumed $b \leq c$. Therefore the only possible numbers are the permutations of $\overline{123}$, of which there are $3! = \boxed{6}$.

26. To have exactly seven factors, the number must be of the form p^6 for some prime p. Minimizing this is equivalent to minimizing p, so we want p to be the smallest prime - 2. Our desired number is thus $2^6 = \boxed{64}$, which indeed has the seven factors 1, 2, 4, 8, 16, 32, and 64.

27. Before any of the operations, $\frac{1}{2}$ of the spinner is black and $\frac{1}{2}$ is white. Alex then changes $\frac{1}{3}$ of the white region, or $\frac{1}{6}$ of the whole spinner, to black, making $\frac{2}{3}$ of the spinner black and $\frac{1}{3}$ of it white. Kelvin then changes $\frac{1}{3}$ of the black region, or $\frac{2}{9}$ of the whole spinner, to white, making $\frac{4}{9}$ of the spinner black and $\frac{5}{9}$ of it white. Our answer is therefore $\boxed{\frac{4}{9}}$.

28. Clearly this computation is infeasible to do directly, so we need to find a more clever method. Let $x = 102$ and $y = 2$. Then

$$102^4 - 8 \cdot 102^3 + 24 \cdot 102^2 - 32 \cdot 102 + 16 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4,$$

which is precisely the expansion of $(x - y)^4$! Therefore our answer is $(102 - 2)^4 = 100^4 = \boxed{100000000}$.

29. There are 10 pages with 1-digit numbers (0 through 9), and 90 pages with 2-digit numbers (10 through 99). This accounts for $1 \cdot 10 + 2 \cdot 90 = 190$ digits, leaving $2014 - 190 = 1824$ digits left to be written. At 3 digits apiece, this accounts for 608 more pages. Thus there are a total of $10 + 90 + 608 = \boxed{708}$ pages.

30. Let O_1, O_2, and O_3 be the centers of the three unit circles, and ABC be the desired equilateral triangle. Dropping altitudes from O_1 and O_2 to $\overline{AB}$, meeting the line at C and D respectively, gives $AB = AC + CD + DB$. Since $O_1C = 1$ and $\angle ABO_2 = \angle BAO_1 = 30°$, we have $AC = DB = \sqrt{3}$. Furthermore, $CD = O_1O_2 = 2$. Thus $AB = 2 + 2\sqrt{3}$, making the area of ABC equal to $\frac{s^2\sqrt{3}}{4} = (\sqrt{3} + 1)^2\sqrt{3} = \boxed{6 + 4\sqrt{3}}$.

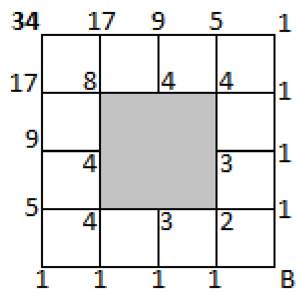
31. The base of $\triangle ABC$ is $AB = 12$, while the height is the altitude from C to AB. This altitude has length at most the radius of the circle, or 6, making the maximum possible area $\frac{12 \cdot 6}{2} = \boxed{36}$.

32. Notice that $x^2 = 2^{64} \implies x = 2^{64 \cdot \frac{1}{2}} = 2^{32}$. Then $x^x = (2^{32})^{2^{32}} = 2^{32 \cdot 2^{32}} = 2^{2^{37}} = 2^y$, making $y = 2^{37}$. Finally, $y = 2^z \implies 2^{37} = 2^z \implies z = \boxed{37}$.

33. Let X and Y be the altitudes from E to AB and G to CD respectively. Then, by the Pythagorean theorem, $EX = GY = \frac{\sqrt{3}}{2}$. Hence $EG = \sqrt{3}+1$. Therefore, the area of $EFGH$ is

$$\frac{EG^2}{2} = \frac{(\sqrt{3}+1)^2}{2} = \boxed{2+\sqrt{3}}.$$

34. Mark each point with the number of paths from it to B. Then, since our only options are to move one unit to the right or one unit down, each number is the sum of the number below it (if any) and the number to the right of it (if any). Clearly, any point on the right or bottom edges has only one possible path - all moves down or all moves right, respectively. The picture shows this process, leading to the answer of $\boxed{34}$.



Alternatively, notice that the only "illegal" paths are the ones passing through the center point, since that is the only one in the middle hole. Hence any combination of 4 moves to the right and 4 moves down, of which there are $\binom{8}{4} = 70$ arrangements, will result in a valid path - unless it passes through the center points, which occurs in $\binom{4}{2}^2 = 36$ of these paths. Hence the answer is $70 - 36 = \boxed{34}$ as before.

35. Currently, Steven has 20 hours of material to catch up. Every Sunday, this amount increases by 1.5 hours (as he watches .5 hours, and two more hours of material are released), and every other day that amount decreases by 0.5 hours. Hence, every full week this amount decreases by 1.5 hours. After 13 such weeks, there is thus just half an hour of material left. Then, on that day (a Sunday), this becomes 2 hours, which takes an additional 4 days to finish. Hence it takes Steven $13 \cdot 7 + 4 = 95$ days to catch up. Noting that October has 31 days, November has 30, and December has 31, Steven will finish exactly $95 - 31 - 30 - 31 = 3$ days after January 12th, which is $\boxed{\text{January 15th, 2015}}$.

36. Let this probability be p. The probability of getting heads exactly twice is $\binom{2015}{2} p^2(1-p)^{2013}$, as we must choose exactly 2 flips to get heads

on and exactly 2013 to get tails on. Similarly, the probability of getting heads exactly three times is $\binom{2015}{3}p^3(1-p)^{2012}$. Equating the two,

$$\binom{2015}{2}p^2(1-p)^{2013} = \binom{2015}{3}p^3(1-p)^{2012}$$

$$\implies \binom{2015}{2}(1-p) = \binom{2015}{3}p$$

$$\implies 1-p = 671p \implies 1 = 672p \implies p = \boxed{\frac{1}{672}}$$

37. Kelvin the Frog can choose any two lilypads to be the left side of his houses, provided that neither of them are at the rightmost end and that they are not adjacent. There are $\binom{41}{2}$ ways to choose 2 lilypads that are both not the rightmost pad, and 40 of these configurations have the two adjacent, making the answer $\binom{41}{2} - 40 = \boxed{780}$.

    Alternatively, if Kelvin makes one house using the leftmost two pads, there are 39 remaining positions that his second house could be in. If Kelvin makes a house using the next leftmost two pads, there are 38 remaining positions that his second house could be in. Continuing in this fashion, there are $39 + 38 + \ldots + 1 = \frac{39 \cdot 40}{2} = \boxed{780}$ possibilities as before.

38. Label the points A, B, C, D, E, F so that $AB = CD = EF = 2$ and $BC = DE = FA = 4$. Extend $\overline{FA}, \overline{BC}$, and $\overline{DE}$ to meet at points X, Y, Z. Then XYZ is an equilateral triangle with side length $4 + 2 + 2 = 8$, and thus has an area of $\frac{8^2\sqrt{3}}{4} = 16\sqrt{3}$, while AXB, CYD, EZF are equilateral triangles with side length 2 and areas of $\sqrt{3}$. Hence the area of $ABCDEF$ is $16\sqrt{3} - \sqrt{3} - \sqrt{3} - \sqrt{3} = \boxed{13\sqrt{3}}$.

39. Let the square be $ABCD$ and suppose that they stand at points A and C. Then the desired region is the intersection of the unit circles centered at A and C. Notice that the quarter-circle ABD has area $\frac{\pi}{4}$, while *triangle* ABD has area $\frac{1}{2}$. Hence the area formed by $\overline{BD}$ and *arc* BD is $\frac{\pi}{4} - \frac{1}{2}$. Twice this area, the desired region, is $\boxed{\frac{\pi}{2} - 1} = \boxed{\frac{\pi - 2}{2}}$.

40. Firstly, notice that AB is perpendicular to O_1O_2 since $O_1A = O_1B$ and $O_2A = O_2B$. Let the intersection of AB and O_1O_2 be X. Then

$$\overline{XC}^2 + \overline{XO_1}^2 + \overline{XB}^2 + \overline{XO_2}^2$$

$$= (\overline{XC}^2 + \overline{XO_1}^2) + (\overline{XB}^2 + \overline{XO_2}^2) = \overline{CO_1}^2 + \overline{BO_2}^2$$

$$= (\overline{XC}^2 + \overline{XO_2}^2) + (\overline{XB}^2 + \overline{XO_1}^2) = \overline{CO_2}^2 + \overline{BO_1}^2$$

hence $CO_1^2 - CO_2^2 = BO_1^2 - BO_2^2 = 5^2 - 3^2 = \boxed{16}$.

Alternatively, we can "cheat" on this problem by assuming the two circles are tangent. Then A, B, C are all the same point, and thus clearly $CO_1^2 - CO_2^2 = 16$. Notice that this is valid because the location of C is irrelevant, as long as it lies on the line AB.

41. Expanding the equation gives

$$f(x)^2 + 2f(x)g(x) + g(x)^2 - f(x)^2 + 2f(x)g(x) - g(x)^2 = 0$$

so $4f(x)g(x) = 0 \implies f(x)g(x) = 0$. Thus either $f(x) = 0$ or $g(x) = 0$. The solutions to the former are $x = 1$ and $x = 5$, while the solutions to the latter are $x = 3$ and $x = 4$. Their sum is $\boxed{13}$. Alternatively, Vieta's formulas tell us that the sum of the 4 roots is $6 + 7 = \boxed{13}$.

42. Notice that the fact that the contructed figures are hexagons is irrelevant, as it is only relevant that the area is directly proportional to the square of the side length - true for any similar polygons. Hence, as long as we are constructing similar shapes on each side, it does not matter which shape we choose. Thus we can assume instead that they are squares, and then the Pythagorean theorem immediately tells us that $P_3 = P_1 + P_2 = 16 + 36 = 52$.

Alternatively, the area of a regular hexagon with side length s is $\frac{s^2\sqrt{3}}{4}$, so we can solve for $AB^2 = \frac{64}{\sqrt{3}}$ and $BC^2 = 48\sqrt{3}$, hence $CA^2 = \frac{208\sqrt{3}}{3}$ and $\frac{CA^2\sqrt{3}}{4} = \boxed{52}$.

43. Let the width of the rectangle be x, so that the length is $12 - 2x$ (as one of the length sides is covered by the barn). Then we need to maximize the quantity $x(12 - 2x)$. Notice that this is equal to $-x(2x - 12) = -2(x^2 - 6x) = -2(x^2 - 6x + 9) + 18 = 18 - 2(x - 3)^2$. Since $(x - 3)^2$ is always nonnegative, the maximum possible value of this expression is $\boxed{18}$, attainable when $x = 3$.

44. Notice that 2014^2 is the square of a four digit number, and hence has at most 8 digits. Thus $s(2014^2)$ is at most $9 \cdot 8 = 72$. As such, $s(s(2014^2))$ is at most 15, and therefore $s(s(s(2014^2)))$ is a single digit number. Furthermore, the sum of the digits of a number is equivalent to the number modulo 9, hence the problem is equivalent to finding the remainder 2014^2 leaves upon division by 9. As 2014 leaves a remainder of 7 upon division by 9, $2014^2 \equiv 7^2 \equiv \boxed{4} \pmod 9$.

45. Since a, b, c are all positive, a^{b-c} is also positive and hence $b > c$. Furthermore, division by a^c gives us $a^{b-c} - 1 = a^{b-2c}$. Since $b > c$, a^{b-c} is an integer, and hence the LHS is an integer. Therefore, so is a^{b-2c}, implying that $b \geq 2c$. If $b - 2c > 0$, then $a \mid a^{b-c} - a^{b-2c} = 1$, implying that $a = 1$ -

a clear contradiction as $1-1 \neq 1$. Hence $b-2c=0$, and thus $a^{b-2c}=1$. Thus $a^{b-c}-1=1 \implies a^{b-c}=2$, possible only if $a=2$ and $b-c=1$. Solving gives us $b=2, c=1$, which is indeed a solution as $2^2-2^1=2^{2-1}$. Hence $a+b+c=\boxed{5}$.

46. The roots of $x^2-17x+k$ sum to 17, and they are both real if $17^2-4k > 0 \implies 289 > 4k \implies 72 \geq k$. Hence our answer is $72 \cdot 17 = \boxed{1224}$.

47. Let G be the centroid of ABC, which is the intersection of AX, BY, and CZ. Furthermore, since these are all medians, $AG = 2XG$ and $BG = 2YG$. Hence $GX' = \frac{5}{2}GA$ and $GY' = \frac{5}{2}GB$, meaning that $[GX'Y'] = \frac{25}{4}[GAB]$. Analogously, $[GY'Z'] = \frac{25}{4}[GBC]$ and $[GZ'X'] = \frac{25}{4}[GCA]$, hence

$$[GX'Y']+[GY'Z']+[GZ'X'] = \frac{25}{4}([GAB]+[GBC]+[GCA]) = \frac{25}{4}[ABC]$$

$$\implies [X'Y'Z]' = \boxed{275}$$

48. Since $\angle ABC$ is right, B lies on the circle with diameter AC, and therefore $MA = MB = MC = 13$. Thus

$$r_1 \cdot \frac{13+13+10}{2} = [ABM] = \frac{1}{2}[ABC] = 60$$

and

$$r_2 \cdot \frac{13+13+24}{2} = [ACM] = \frac{1}{2}[ABC] = 60$$

Hence $r_1 r_2 = \frac{120^2}{36 \cdot 50} = \boxed{8}$.

49. Once Kelvin flips a tail, he will go to sleep immediately after flipping a head. Hence, any valid string will be of the form "(HH...HT)(TT...TH)", where an H denotes a head and a T denotes a tail. The expected length of this string is our answer. By linearity of expectation, this is equivalent to the sum of the expected length of an "HH...HT" string and the expected length of a "TT...TH" string, which are of course equal by symmetry. Let E be this expected length. Then there is a $\frac{1}{2}$ chance that the string is of length 1 (flipping a tail in the first case, or a head in the second), and a $\frac{1}{2}$ chance that the string is of expected length $E+1$ (flipping a head in the first case, or a tail in the second). Hence $E = \frac{1}{2}E + 1 \implies E = 2$, and thus our answer is $2 \cdot 2 = \boxed{4}$. Note that we can also find the value of E by finding

$$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \dots$$

$$= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) + \left(\frac{1}{4} + \frac{1}{8} + \dots\right) + \dots$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{1}{4}}{1-\frac{1}{2}} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \dots$$
$$= \frac{1}{1 - \frac{1}{2}} = 2,$$

but clearly the other method is significantly easier.

50. AJ bought 2015 houses, each at a point $(2014, k)$ with $0 \le k \le 2014$. Such a house will be safe only if there exists a point (x, y) with $0 < x, y < 2014$ such that $(0,0), (x, y)$, and $(2014, k)$ are collinear, meaning that the slope between $(0,0)$ and (x, y), $\frac{y}{x}$, is equal to the slope between $(0,0)$ and $(2014, k)$, $\frac{k}{2014}$. This is possible only when k and 2014 are *not* relatively prime, i.e. there exists some $d > 1$ such that $d \mid k$ and $d \mid 2014$. Then we can take $(x, y) = (\frac{2014}{d}, \frac{k}{d})$, and AJ's house will survive.

    We thus need to count the number of k such that $\gcd(k, 2014) \neq 1$. Notice that, since everything divides 0, $k = 0$ is one of these such numbers. Other than that, if $d \mid 2014$, either $2 \mid d, 19 \mid d, 53 \mid d$, or some combination of the three. Hence, by PIE, there are

$$\frac{2014}{2} + \frac{2014}{19} + \frac{2014}{53} - \frac{2014}{2 \cdot 19} - \frac{2014}{2 \cdot 53} - \frac{2014}{19 \cdot 53} + \frac{2014}{2 \cdot 19 \cdot 53} = 1078$$

    k between 1 and 2014 such that $\gcd(k, 2014) \neq 1$, and accounting for $k = 0$ gives our final answer of $\boxed{1079}$.

    Alternatively, but much more advanced, there are $\phi(2014) = 1 \cdot 18 \cdot 52 = 936$ k such that $\gcd(k, 2014) = 1$, meaning that there are 936 houses that will be destroyed and thus $2015 - 936 = \boxed{1079}$ houses that will stay intact.