2015 Joe Holbrook Memorial Math Competition
7th Grade Exam

The Bergen County Academies Math Team

October 11th, 2015

Instructions

DO NOT TURN OVER THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

The Joe Holbrook Memorial Math Competition is a 90-minute, 50-question exam. Each question has exactly one correct answer; writing two different answers, EVEN IF ONE IS CORRECT, is worth no credit. In addition, **only answers written on the answer sheet provided will be graded.**

Be advised that proctors cannot answer any questions about terminology, notation or the questions.

On the JHMMC, you may only use writing utensils, erasers, and scrap paper provided by the proctors. You MAY NOT use calculators, compasses, protractors, straightedges, or your own scrap paper.

All answers must be fully simplified and exact. For example:

- Write $2\sqrt{2}$, rather than $2.83\ldots$
- Write $\frac{4}{3}$, rather than $1\frac{1}{3}$ or $1.33\ldots$

In addition, know that $[A_1A_2\cdots A_n]$ denotes the area of n-gon $A_1A_2\cdots A_n$.

1. What is $5 + 7 + 7 + 9 + 9 + 5 + 3 + 2 + 2 + 1$?

2. What is the remainder when 123456 is divided by 5?

3. How many times does the Sun set between noon on Saturday and noon on the following Thursday?

4. Find the 100th member of the following sequence: $1, 4, 7, 10, \ldots$

5. Sung Hyup was fifteen years old four years ago. Sung Hyun is three years younger than him. How old will Sung Hyun be in two years?

6. What is $0 - 1 + 2 - 3 + 4 - 5 + \cdots - 99 + 100$?

7. How many multiples of 17 are positive three-digit integers?

8. Minsung is taking a forty-question test. He answered questions 3 through 14 correctly, as well as questions 17 through 35, but nothing else. If all questions were answered either correctly or incorrectly, how many questions did he get right?

9. Each weekday, Young Guy receives a certain amount of stickers from his teacher, Dr. Lal, based on his behavior. If he behaves well, he gets 1, if he behaves poorly, he gets 0, and if he behaves amazingly (!) he gets 2. This goes on for 3 weeks exactly. Given that Young Guy behaves well on Mondays and Tuesdays, amazingly on Wednesdays, and poorly on Thursdays and Fridays, how many stickers will he have at the end of the 3 weeks to show off to his friend Zack?

10. Jungle Jim, the proprietor of the jungle gym, is charging admission. If the admission fee for Matt is \$20, but he has a coupon for a 10% discount, and the admission fee for Tanny is \$25, but he has a coupon for a 15% discount on every dollar after \$5 that he pays, who pays more, and how much does he pay?

11. Jungle Jim is now actively preventing people from entering the jungle gym! To enter, Zack and Erik now have to scale and descend two completely vertical walls in succession. They start at sea level, and climb over a wall that is 10 feet high and 0.5 feet thick, which leads to an area of elevation 2 feet. They go forward 25 feet, and climb over a wall that is 11 feet high (relative to sea level) and 1.5 feet thick. This leads to an area which is at elevation 1 foot above sea level. They go forward 10 feet, and have reached the jungle gym! What is the total length of their path, in feet?

12. What is the sum of the number of faces and edges of a pyramid with an octogonal base?

13. Kelvin the Frog is having trouble finding his way around college. He knows that the cafeteria is 20 hops away from his dorm, while the gym is 10 hops away. What is the difference, in hops, between the longest possible distance from the cafeteria to the gym and the shortest possible distance?

14. What is the smallest positive integer divisible by 4, 5, and 6?

15. If $a + b = 7$ and $a^2 + b^2 = 29$, what is ab?

16. Two sides of a triangle have lengths 12 and 13. What is the largest possible integer side length of the third side?

17. What is the last digit of 2^{245}?

18. Let $x \mho y$ be defined as $8x - 3y - 7$. Compute $((7 \mho 13) \mho 23)$.

19. Rectangle $ABCD$ has width 3 and length 6. What is the length of AC?

20. Find the difference between the degree measure of an internal angle of a regular octagon and an external angle of a regular decagon.

21. How many distinct ways are there to arrange the letters in the word JHMMC?

22. A circle is inscribed in a square, which is inscribed in a circle, which is inscribed in a square. The smaller circle has radius 1. Color red the regions inside the larger circle that are outside the smaller square, and color red as well the interior of the smaller circle. In terms of π, what is the total area that is colored red?

23. What is the remainder when 5^{2015} is divided by 24?

24. Define a sequence of integers as follows: $a_0 = 0, a_1 = 1, a_2 = 2$ and for $n \geq 3$, $a_n = a_0 + \cdots + a_{n-1}$. Find the value of a_8.

25. What is the side length of a cube whose volume is numerically twice its surface area?

26. What is the largest n such that $1^1 + 2^2 + \cdots + n^n \leq 50000$?

27. The 400-digit number 1234567812345678 . . . 12345678 is written on a piece of paper. June then repeatedly erases every 7th digit of the number. When he runs out of digits to erase, he begins this erasing process over again and again, starting from the beginning of the remaining number. At the end of this process, a six-digit number remains. What is this number?

28. Points E and F are on side AB of square $ABCD$ such that $EF = 4$ and E is closer to A than B. If $AB = 6$, what is $[EFCD]$?

29. Compute the largest prime factor of $6^{12} + 12^6$.

30. Kang Myung has already written 30 lines of code to be submitted to Dr. Nevard. However, he refuses to hand in code unless the number of lines he has written is a multiple of 17 and it is 11:57 PM. He writes exactly 6 lines of code each morning. If it is currently noon three days before the deadline, how many days late will he submit his code?

31. A coin lands on heads with likelihood $\frac{2}{3}$ and tails with likelihood $\frac{1}{3}$. If Ryan the ⫛ tosses the coin 4 times, what is the probability that Ryan the ⫛ will see at least two heads?

32. Evaluate $1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + 25)$.

33. Real numbers $a - 7$, $b + 4$, and $c + 6$ form a geometric sequence, and numbers a, b and c lie in an arithmetic sequence and have an arithmetic mean of 46. What is the arithmetic mean of the terms in the geometric sequence?

34. If the sum of the digits of a 3-digit number is 15, and the number itself is a multiple of 35, what is this number?

35. Let $a = 3^{2015}$, $b = 2015^3$, $c = 1009^{1009}$. Order the three numbers from largest to smallest, using the variable names.

36. Three faces of a rectangular prism have areas 16, 20, and 45. What is the volume of this prism?

37. A has coordinate $(-1, 3)$, B has coordinate $(2, 7)$, and C has coordinate $(c, 0)$. What is the greatest possible value of $|AC - BC|$?

38. A list of digits is created in the following way: 1 is listed 1 time, followed by 2 being listed 2 times, followed by 3 being listed 3 times, etc., and in general, n being listed n times. What digit occupies the 2015th position on this list?

39. There are six cards, labeled $2, 2, 4, 4, 6, 6$. What is the probability that if Zack picks three cards, without replacement, he can make a triangle with side lengths given by the numbers on the selected cards?

40. Compute the smallest positive integer with at least 20 positive integer factors.

41. Suppose a and b are the roots of the quadratic $P(x) = x^2 - 52x + 365$. Let $Q(x) = x^2 - (ab)x + (a+b)$, and let c and d be solutions to the equation $Q(x) = 0$. Find $a+b+c+d$.

42. In right triangle ABC, $\angle C = 90°$, $\angle A = 60°$, and $AC = 1$. D is on BC, E is on AB, so that $\angle ADE = 90°$ and $AD = DE$. What is the length of segment BE?

43. If $x^2 - 13x + 1 = 0$, what is the unit digit of $x^4 + x^{-4}$?

44. Compute the infinite sum $\frac{4}{3} + \frac{7}{9} + \frac{10}{27} + \frac{13}{81} + \cdots$

45. A nonagon is constructed as follows: Starting with vertex A_1, eight vertices $A_2, \ldots, A_9$ are made in succession such that $A_1A_2 = 1, A_2A_3 = 2, \ldots, A_8A_9 = 8$ and two consecutive sides form a $135°$ angle, and the sides spiral out from A_1. If $A_9A_1 = \sqrt{a + b\sqrt{2}}$, find $a - b$.

46. Jen is looking at 12-hour analog clock whose hour hand and minute hand are of lengths 2 and 3 respectively. If Jen looks at the clock at a random time between 12 PM and 1 PM, what is the probability that the distance between the tips of the hour and minute hands is less than or equal to $\sqrt{7}$?

47. $Z(x)$ is a polynomial such that $Z(1) = 2$, $Z(2) = 12$, and $Z(3) = 28$. The remainder of $\frac{Z(x)}{(x-1)(x-2)(x-3)}$ can be expressed as $ax^2 + bx + c$. What is $(ab)^c$?

48. What is the expected area of the triangle formed by the origin, the point $(10, 0)$, and a random point on the triangle defined by the points $(2, 3)$, $(26, 3)$, and $(14, 19)$?

49. Real numbers a, b, and c satisfy these conditions: $a + b + c = 1$, and

$$\frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{c+a-b} = 1.$$

What is the value of abc?

50. Let $f(x) = \frac{x^2+4x+13}{3x^2+2x+3}$, M be the maximum of f over all reals, and m be the minimum of f over all reals. Find the product of M and m.