Joe Holbrook Memorial Math Competition

6th Grade Solutions

October 9th, 2016

1. The list of numbers consists of integers between -17 and 17, which includes 0. Any product of a number and 0 is $\boxed{0}$.

2. $2, 4,$ and 6 are the even numbers on a regular six-sided die. Because each number has an equal likelihood of being rolled, the probability of rolling an even number is $\frac{3}{6} = \boxed{\frac{1}{2}}$.

3. There are twelve inches per foot, so Yousun is $5 * 12 = 60$ inches tall. Youjung is therefore $60 + 6 = \boxed{66}$ inches tall.

4. The phone has a maximum battery life of $10 + 60 = 600$ minutes. Therefore, the phone has $12\% \cdot 600 = \boxed{72}$ minutes left.

5. The prime factorization of 2016 is $2^5 \cdot 3^2 \cdot 7$. The prime factorization of 2772 is $2^2 \cdot 3^2 \cdot 7 \cdot 11$. The greatest common factor can be found by identifying the least exponent of each prime factor: $2^2 \cdot 3^2 \cdot 7 = \boxed{252}$.

6. Aligning our multiplicands in vertical fashion, we see that many numerators and denominators cancel, leaving a final answer of $\boxed{\frac{1}{6}}$.

7. $2^1 1 = 2048$, and $2^1 0 = 1024$ Therefore, the smallest n satisfying the equation is $\boxed{11}$.

8. His cake recipe calls for $1.5 + 1 = 2.5$ cups of butter and milk per cake. If he triples his recipe, then the total amount of butter and milk he will use is $2.5 \cdot 3 = \boxed{7.5}$ cups.

9. Let's call our number n. We then perform many operations: first we obtain $n + 2016$, then $4 \cdot (n + 2016)$, then $4 \cdot (n + 2016) - 12$, then $\frac{4 \cdot (n + 2016) - 12}{4} = n + 2016 - 3 = n + 2013$, then $(n + 2013) - n = \boxed{2013}$.

10. Since doubling any number gives us an even number, we have to work backwards. On Friday, Kelvin had $\frac{48}{2} = 24$ lilypads. On Thursday, he had $\frac{24}{2} = 12$ lilypads. On Wednesday, he had $\frac{12}{2} = 6$ lilypads. On Tuesday, he had $\frac{6}{2} = 3$ lilypads, making our day $\boxed{\text{Tuesday}}$.

11. One liter is also 1000 milliliters. One liter of the smoothe has $10\% \cdot 1000 = 100$ milliliters of water, and one liter of the soda has $2\% \cdot 1000 = 20$ milliliters of water. Therefore, when mixed, we have $100 + 20 = 120$ milliliters of water, and a total of $1000 + 1000 = 2000$ milliliters of liquid. So, $\frac{120}{2000} = \frac{6}{100} = \boxed{6\%}$.

12. If Arthur wants to run 13 miles at an average speed of 10 mph, he needs to complete the 13 miles in $\frac{13}{10} = 1.3$ hours, or $1.3 \cdot 60 = 78$ minutes. He has run 6 miles, so he has $13 - 6 = 7$ miles left to run. He has only $78 - 50 = 28$ minutes, or $\frac{28}{60} = \frac{7}{15}$ hours to do so. Therefore, his speed is $\frac{7}{\frac{7}{15}} = \frac{7 \cdot 15}{7} = \boxed{15}$ miles per hour. Every 7 days after a Friday is also a Friday. Therefore, if you divide 2016 by 7, the quotient is irrelevant, and the remainder tells you how many days of the week you must count before reaching the correct day of the week. However, since 2016 is perfectly divisible by 7, the remainder is 0, so the day of the week will be the same. Therefore, the 2016th day after January 1st, 2016 will be on a $\boxed{\text{Friday}}$.

13. Notice that 100^3 is exactly equal to 1,000,000. Therefore, cubing any integer less than 100 will result in a number that is less than 1,000,000. The largest integer less than 100 is $\boxed{99}$.

14. Let a slice of plain pizza cost x dollars and a slice of pepperoni pizza cost $x + 0.5$ dollars. David and June ordered 3+2= 5 slices of plain pizza and $2 + 4 = 6$ slices of pepperoni pizza, so $5x + 6(x + 0.5) = 11x + 3 = 25$. Since $x = 2$ from the previous equation, a slice of plain pizza costs \$2 and a slice of pepperoni pizza costs \$2.50. David ordered 3 slices of plain pizza and 2 slices of pepperoni pizza, so he paid $3 \cdot 2 + 2 \cdot 2.5 = 6 + 5 = \boxed{11}$ dollars.

15. The value of $f(\pi^2)$ is simply $\pi^2 + 1$, and $g(\pi^2 + 1) = \lfloor \pi^2 + 1 \rfloor = 10$, as π^2 lies between 9 and 10. Thus, $h(\pi^2) = \boxed{10}$.

16. There are 3 different choices for buying milk, 2 different choices for buying eggs, and 4 different choices for buying butter, so the total number of ways to buy one of each is equal to $3 * 2 * 4 = \boxed{24}$.

17. 54 flips $= 18 \cdot 5 = 90$ flops. 90 flops $= 10 \cdot 14 = \boxed{140}$ flaps.

18. Andrew and Dennis finish 3% in 1 hour, so the other three will do 15% in one hour. There is 97% left, so the answer is simply $\frac{97}{15} \cdot 60 = \boxed{388}$.

19. Arthur ran 40 meters in the first 5 seconds. He only has to run for $\frac{100 - 40}{3} = \frac{60}{3} = 20$ more seconds. Sunny ran for 32 meters in the first 8 seconds. That means that in $25 - 8 = 17$ seconds, he must run 68 meters, which is an average speed of $\frac{68}{17} = \boxed{4}$ m/s.

20. Note that 2^4 has a units digit of 6. Since $2^{2016} = (2^4)^5 04 = 6^{504}$, and every power of 6 ends in 6, we know 2^{2016} has a units digit of 6. Also note that $3^4 = 81$ has a units digit of 1. Since $3^{2016} = (3^4)^{504} = 81^{504}$, we know 3^{2016} has a units digit of 1. Our answer is therefore $1 + 6 = \boxed{7}$.

21. Using the common area formulas for both figures: The square's side s is the solution to $s^2 = 25$, and is therefore 5. For the equilateral triangle, $\frac{s^2 \cdot \sqrt{3}}{4} = 9\sqrt{3}$, so the side length is 6. The difference between the two is $6 - 5 = \boxed{1}$.

22. Recall that $2015 = 5 \cdot 13 \cdot 31$, $2016 = 2^5 \cdot 3^2 \cdot 7$, and 2017 is prime. The number of factors of a positive integer n with prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$. Thus, $A = 2 \cdot 2 \cdot 2 = 8$, $B = 6 \cdot 3 \cdot 2 = 36$, and $C = 2$, with average value $\frac{A + B + C}{3} = \boxed{\frac{46}{3}}$.

23. For $n < 7$, $n! < 1000$, and thus, we must start our search from $n = 7$. Recall that $7! = 5040$ and $7^3 = 343$, so that $7! - 7^3 = 5040 - 343 = 4697 > 2016$. Hence, the least such n is $\boxed{7}$.

24. The resulting figure is a semicircle with radius $\sqrt{2}$, and two half-squares with side length 1, which yields an area of $\boxed{\pi + 1}$.

25. "The largest number can be constructed by selecting 9, then 6 (since 7 is also odd, and 8 is too close to 9), then 9, then 6, then 9 again.

    Similarly, the smallest legal number is 14141. Thus the sum is $\boxed{111110}$.

26. If he makes x more throws, then his shooting percentage will be $\frac{1 + x}{48 + x}$. We want this value to equal $\frac{6}{100}$. Solve the proportion for x to get $x = \boxed{2}$.

27. The ratio of the areas is $\frac{360}{40} = 9$, hence the ratio of the sides will be $\sqrt{9} = 3$. The length h of the larger hypotenuse will satisfy $\frac{h}{15} = 3$, and we find $h = \boxed{45}$.

28. For a house to be occupied by only 1 cat or 1 human, the house number should only be divisible by 7 or only be divisible by 3. If the house number is divisible by both, both a cat and human will live in the house. The sum of the houses with cats living in the house is $3 \cdot (1 + 2 + 3 + \ldots + 25 + 26)$. This comes out to $\frac{26 \cdot 27}{2} \cdot 3 = 1{,}053$. The sum of the houses with humans living in the house is $7 \cdot (1 + 2 + 3 + \ldots + 10 + 11)$. The sum of the houses for humans is $\frac{11 \cdot 12}{2} \cdot 7 = 462$. The houses that will have both cats and humans will be divisible by 21 as the LCM of 7 and 3 is 21. The sum of the houses with numbers divisible by 21 is $21 \cdot (1 + 2 + 3)$. This comes out to $21 \cdot (6) = 126$. Therefore, the sum of the houses that have only one cat or one human in it is $1053 + 462 - (2 * 126) = \boxed{1{,}263}$.

29. If $x \le \frac{1}{2}$, then the equation becomes $\frac{1}{2x} = \frac{1}{x+\frac{1}{2}}$, yielding $x = \frac{1}{2}$. If $\frac{1}{2} \le x \le 1$, then $\frac{1}{2x} = x + \frac{1}{2}$, yielding $x = \frac{1}{2}$ again. Finally, if $x \ge 1$, then $\frac{x}{2} = x + \frac{1}{2}$, and there is no positive solution. Thus, the only solution is $x = \frac{1}{2}$, so the answer is $\boxed{\frac{1}{2}}$.

30. Let's number the positions from left to right with 1 through 5. The two M's can either be in 1 and 2, 1 and 5, or 2 and 5. For each of these 3 cases, exactly 1 of the remaining 3 letters (J, H, and C) has its original position as an available position. As a result, that letter has 2 choices to be put. Then, the remaining 2 letters can be placed in the remaining 2 positions in any order, since their original positions were taken by the two M's. So, there are $3 \cdot 2 \cdot 2 = \boxed{12}$ total derangements.

31. The net rate of water flow into the tank is $4 - 2.5 = 1.5$ liters per minute. Thus, it will take $\frac{150}{1.5} = 100$ minutes to fill the tank, or, equivalently, $\boxed{6000}$ seconds.

32. From the given averages, we have $a+b+c+d = 28$ and $d+e+f+g = 16$. We also have $a+b+c+d+e+f+g = 35$. Adding the first two equations we have $a+b+c+2d+e+f+g = 44$, and subtracting the third from this gives us $d = \boxed{9}$.

33. In the first 2 days, exactly 20% of the first wall is built. That leaves 4.8 walls left. Typically, it would take 10 workers 48 days to build 4.8 walls. Since there are only 8 days left, the workers must do 6 times as much to get the same work done in $\frac{8}{48} = \frac{1}{6}$ of the time. The workers can not work faster, so that means there must be 6 times as many workers, making the final amount 60. We are looking for the amount added, so the answer will thus be $60 - 10 = \boxed{50}$.

34. At first the restrictions may seem daunting, but because the seating arrangements have so many limitations, it is actually fairly easy to just count them out. We can start off by placing Ben and his friends. Ben can either sit on one of the ends of the row or in the middle seat. He cannot sit in the second or fourth seat because he must have both friends immediately adjacent to him, which would force Andrew and Dennis together. So, consider the case where Ben is on the end. And example of this would be BCAED. We can either choose to put Caleb or Eddie in the second seat, leaving the other to sit between Andrew and Dennis. Furthermore, we can swap the positions of those two, so this case altogether has 4 arrangements. However, we can also reflect each arrangement so that Ben is sitting on the rightmost seat, giving us another 4 arrangements. We could also have Ben sitting in the middle seat. In this case, Caleb and Eddie must be occupying the second and fourth seats; we can accomplish this two ways (CBE or EBC). Andrew and Dennis are left with the two remaining seats and there are two ways to place them into these. Therefore, there are also 4 arrangements in this case. We can get the total number of ways by adding these cases together, so we have $4 + 4 + 4 = \boxed{12}$ ways.

35. If we add the two equations, we get that $2 \cdot (x+y)^4 = 1 + 3 = 4$, so $(x+y)^4 = \frac{4}{2} = 2$, so $(x+y) = \pm 2^{\frac{1}{4}}$. Since we are asking for the absolute value, take the positive answer of $\boxed{2^{\frac{1}{4}}}$.

36. The probability of the test being positive is

$$\frac{1}{150} \cdot \frac{96}{100} + \frac{149}{150} \cdot \frac{4}{100} = \frac{692}{15000}.$$

The probability of a person having the virus and testing positive is

$$\frac{1}{150} \cdot \frac{96}{100} = \frac{96}{15000}.$$

Thus, the probability that Zack has the muggy virus given that he tested positive is $\frac{\frac{96}{15000}}{\frac{692}{15000}} = \boxed{\frac{24}{173}}$.

37. Let a, b, c be the number of \$10,\$30,\$40 shirts he bought respectively. We have that $10a + 30b + 40c = 240$, which becomes $a + 3b + 4c = 24$, and we have $a + b + c = 12$. We subtract through to get $2b + 3c = 12$. But c cannot be 0 because he bought at least one of each type, and likewise c cannot be 4 because then b would be 0. Similarly, c can't be 1 or 3 because b would not be an integer. Then the only possible value of c is $\boxed{2}$, and that is our answer.

38. There are two areas that can be considered here: ones where the chain is not restricted by any vertices, and those where the chain is. When the chain isn't restricted, Bessie can roam freely, forming the sector of a circle. Since regular hexagons have angles of $120°$, Bessie can roam $\frac{360-120}{360} = \frac{2}{3}$ of an entire circle with radius 20. This is $\frac{800\pi}{3}$. However, at the adjacent vertices, she can bend the chain. At this point, she has $20 - 15 = 5$ feet of chain left. The sector would be only $60°$, since the angles are supplementary to $120°$. The area of this smaller region is $\frac{25\pi}{6}$. There are two, so the total area is $\frac{25\pi}{3}$. The combined area is thus $\frac{800+25\pi}{3} = \boxed{275\pi}$.

39. The bird flies for exactly $\frac{1000}{10+10} = 50$ seconds. At 12 meters per second, the bird travels a total distance of $50 \cdot 12 = \boxed{600}$ meters.

40. Once Arthur decides how many red and yellow dumplings he will eat, the number of green dumplings becomes automatically determined. Also, he can eat *any* number of red or yellow dumplings, since the total number of red or yellow dumplings is 5, which is less than 6. Therefore, he has 4 choices for the number of red dumplings (0, 1, 2, or 3) and 3 choices for the number of yellow dumplings (0, 1, or 2), for a total of $4 \cdot 3 = \boxed{12}$ choices.

41. This problem is just a stars and bars problem with the stars being the cat toys and the cats being the bars. In the placement of the bars, there has to be 8 bars to separate the stars as there must be a positive number of stars between each bar. There would be 3 gaps between the stars. Thus there has to be $\binom{8}{3} = \boxed{56}$ possible ways to distribute the identical cat toys to the different cats.

42. Note that $56^3 = (7 + 24 + 25)^3$. Then, we can write the given expression as $56^3 - 7^3 - 24^3 - 25^3 = (7 + 24 + 25)^3 - 7^3 - 24^3 - 25^3$. We recognize that the is expression is of the form $(a + b + c)^3 - a^3 - b^3 - c^3$, which factors into $3(a + b)(a + c)(b + c)$. Thus, we want the largest prime factor of $3(7 + 24)(7 + 25)(24 + 25) = 3(31)(32)(49)$, which is $\boxed{31}$.

43. Factoring out the 7 and rationalizing the denominators yields $7(\frac{\sqrt{2}-\sqrt{1}}{2-1} + \frac{\sqrt{3}-\sqrt{2}}{3-2} + \cdots + \frac{\sqrt{49}-\sqrt{48}}{49-48}) = 7(\sqrt{49} - \sqrt{1}) = \boxed{42}$.

44. There's nothing that makes any single letter greater than the other, so suppose they are all the same. Then $w^2 = \frac{324}{4} \to w = x = y = z = 9$, and $w + x + y + z = 36$. If we want to make this sum greater, we must increase a letter by more than we decrease a letter. However, $|n+1|^2 - n^2 = 2n+1 > 2n-1 = |n-1|^2$. Thus such an increase is impossible, and the greatest possible value of $w + x + y + z$ is $\boxed{36}$.

45. A more rigorous solution is available by the *Cauchy-Schwarz inequality*: $(w^2 + x^2 + y^2 + z^2)(1 + 1 + 1 + 1) \geq (w+x+y+z)^2$, so $324 \cdot 4 \geq (w+x+y+z)^2$. Taking the square root of both sides gives us $18 \cdot 2 = 36 \geq w+x+y+z$. The equation $w + x + y + z = 36$ for $w = x = y = z = 9$, so $\boxed{36}$ is our answer.

46. To get the shortest path, reflect $(18, 4)$ over the line $y = 6$ to get point $(18, 8)$. Then reflect it again over the x-axis to get point $(18, -8)$. The shortest path has the same length as the line segment connecting $(0, 4)$ to $(18, -8)$ and we can use the formula for distance to find that the answer is $\boxed{6\sqrt{13}}$.

47. We consider two cases: when both Anna and Savannah are next to Hannah, and when exactly one of them is next to Hannah.

    Case 1: Anna and Savannah are both next to Hannah, so there are 5! ways to arrange the rest of the people in the other five seats. Then, we multiply by two, since Anna can either be to Hannah's left or Hannah's right, and Savannah will take the remaining seat. Thus, for this case we have $2 \cdot 5!$ arrangements.

    Case 2: Exactly one of Anna and Savannah is next to Hannah. If Anna is next to Hannah, she can either be to Hannah's left or Hannah's right, so she has two choices. Next, Savannah has four choices of where to sit so that she is not next to Hannah or Anna. Finally, there are 5! arrangements for the remaining five people. We can use the same argument for when only Savannah is next to Hannah, so we multiply our product by two. This yields a total of $2 \cdot 2 \cdot 4 \cdot 5!$ arrangements for this case. Therefore, there are a total of $2 \cdot 5! + 2 \cdot 2 \cdot 4 \cdot 5! = \boxed{2160}$ arrangements for both Hannah and Anna to be happy.

48. If N is expanded, there are only two terms, 2016^{2015} and 2016^{2014}. Therefore the base-2016 expression of N is of the form $110\ldots000$. There are $\boxed{2}$ ones in the representation.

49. $(x+y+z)^2 \geqslant 0$, so expansion gives $x^2+y^2+z^2+2\cdot(xy+xz+yz) \geqslant 0$. Plugging in the givens and solving the pair-wise products gives $\boxed{\frac{-9}{2}}$.

50. An important concept used to solve this problem is *recursion*, where one term of a sequence is determined by the previous terms of that sequence. Let P_r be the probability that a red yarn ball and all of its descendants become blue; we similarly define P_y and P_b. Now we look at the behaviour of each colour ball. A red ball has probability $\frac{5}{12}$ of becoming two red balls, $\frac{1}{3}$ of becoming one red ball and one yellow ball, and $\frac{1}{4}$ of becoming one blue ball. Because balls are independent, we can multiply their probabilities of locking into blue. Moreover, The probability that a certain initial state will lock into blue should be the same as the weighted sum of the probabilities that all of its descendent states will lock into blue. Thus we have $P_r = \frac{5}{12}P_r^2 + \frac{1}{3}P_rP_y + \frac{1}{4}P_b$. Similarly, $P_y = \frac{1}{2}P_y^2 + \frac{1}{4}P_y + \frac{1}{4}P_b$. Blue always stays blue, so $P_b = 1$. Evaluating this system of equations for P_r produces the extraneous solution $P_r = 1$ and the solution $P_r = \boxed{1-\frac{\sqrt{10}}{5}}$.