8th Grade Solutions

1. 23 is prime, so 1 and 23 are the only positive integers dividing 23 evenly. Hence, there are **2** positive integers that evenly divide 23.

2. Jie, David and Ben all start with x buttons. Then Jie gives 12 buttons to David and he now has $x-12$ buttons. David has $2(x-12)-6$ buttons. After Ben gives y buttons to David he has $x-y$ buttons and David has $2(x-12)-6+y$, or $2x-30+y$ buttons. David has 20 more buttons than Ben so $2x-30+y-20=x-y$ or $x+2y=50$. If Ben had y less buttons and Jie had y more buttons, they would have the same amount of buttons so $x-12+y=x-2y$ and $y=4$. Solve for x to get $x=42$ and David had $2(42)-30+(4)=\mathbf{58}$ buttons in the end.

3. 0 is included in this product (the product is equal to $(-2010)\times(-2009)\times\cdots\times(-1)\times 0\times 1\times\cdots\times 2010$), so the entire product is **0**.

4. In a deck of regular cards, there are 52 cards. Each deck contains 26 red cards. Of those red cards, 6 of them are royal cards. By using the standard probability formula, we can construct the equation (6 desired cards)/(52 total cards). Simplifying gives us $\mathbf{\frac{3}{26}}$.

5. Because $\overline{AB}+\overline{BC}=\overline{CA}$, A, B, and C lie on a line. Since M is the midpoint of $\overline{AC}$, the distance from M to A is 2.5. The distance from B to A is 2, so the distance from M to B is $2.5-2=\mathbf{0.5}$.

6. There are three H's and two A's, and the tiles are selected without replacement, so the probability is $\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}\times\frac{1}{2}\times\frac{1}{1}=\mathbf{\frac{1}{10}}$.

7. First, find the times, T_1 and T_2 that each of the portions of the trip took to complete: 50 miles $=T_1\times 60$ miles/hour, which implies that $T_1=\frac{5}{6}$ hours. 50 miles $=T_2\times 30$ miles/hour, which implies that $T_2=\frac{5}{3}$ hours. So the total time for the trip is $\frac{5}{6}+\frac{5}{3}=\frac{15}{6}$ hours, and the average speed for the trip is Distance/Time $=\frac{100}{15/6}$ $=\mathbf{40}$ miles/hour.

8. $\log_3 27=\log_3(3^3)=\mathbf{3}$.

9. $64=1\times 6^2+4\times 6^1+4\times 6^0=\mathbf{144}$.

10. $57=2\times 3^3+0\times 3^2+1\times 3^1+0\times 3^0=\mathbf{2010}$.

11. The nth term of the sequence is given by the formula $5n$. Term 1 to term 20 of the sequence sum to $5+10+\cdots+95+100=\left(\frac{\text{First+Last}}{2}\right)\times$Number of numbers being added $=\frac{5+100}{2}\times 20=105\times 10=\mathbf{1050}$. OR The sum of the first 20 terms of the sequence $5,10,15\ldots$ is equal to five times the sum of the first 20 terms of the sequence $1,2,3\cdots$. The latter sequence is equal to $\frac{20\times 21}{2}$, so the former is equal to $5\times\frac{20\times 21}{2}=\mathbf{1050}$.

12. We see that the ones digit for powers of 3 repeat in the following fashion: $3,9,7,1,3,9,7,1,3,\ldots$ since $(3^1)=3$, $(3^2)=9$, $(3^3)=27$, $(3^4)=81$, $(3^5)=243$, etc. We want to find the ones digit for 3^{30}. Looking at the pattern shown above, we see that the ones digits repeat every 4 times. 30 divided by 4 has remainder of 2 so that means ones digit for 3^{30} is the second term of the pattern which is **9**.

13. $x^2\geq 0$ since x is a real number, and $-|y^2-1|\leq 0$, since the absolute value of any number is nonnegative. Hence, $0\leq x^2=-|y^2-1|\leq 0$, so $x=0$ and $|y^2-1|=0$. Hence, $x^{2009}=0$ and $y^2=1$, so $y^{2010}=1$, so our final answer is $0+1=\mathbf{1}$.

14. First of all, observe that $\angle B$ is 120° since $ABCDEF$ is a regular hexagon. Now, triangle ABC is isosceles since $\overline{AB}$ and $\overline{BC}$ are the same lengths. Therefore, $\angle ABC=\angle ACB=30°$. If the triangle was cut in half, it will result in two 30,60,90 triangles leg to leg. Thus, the ratio if $\overline{AC}$ to $\overline{AB}$ is twice the ratio of the long leg to the hypotenuse of a 30,60,90 triangle. The ratio of the long leg to the hypotenuse is $\frac{\sqrt{3}}{2}$. Hence, the answer is $\mathbf{\sqrt{3}}$.

15. The desired probability is the probability that the problem he got wrong is among the last 5 questions. As there are 15 questions, the probability is $\frac{5}{15}=\mathbf{\frac{1}{3}}$.

16. Let $x = 2^a 5^b n$, where n is an integer not divisible by 2 or 5. Since we want the smallest x satisfying the condition, we may let $n = 1$. Since $4x = 2^{a+2}5^b$ is a perfect cube, $a + 2$ and b must be multiples of 3. Since $5x = 2^a 5^{b+1}$ is a perfect square, a and $b + 1$ must be even. Since we wish to make x as small as possible, we wish to make a and b as small as possible. The smallest nonnegative integer a such that a is even and $a + 2$ is a multiple of 3 is 4, and the smallest nonnegative integer b such that b is a multiple of 3 and $b + 1$ is even is 3. Hence, the smallest possible x is $2^4 5^3 = \mathbf{2000}$.

17. $f(ab) = 10a + b - (10b + a) = 10a + b - 10b - a = 9a - 9b = 9(a - b)$. Hence, 9 divides $f(10), f(11), \cdots, f(99)$, 9 is less than greatest common factor of these numbers. On the other hand, $f(10) = 9$, so the greatest common factor of these numbers is not more than 9, so our answer is **9**.

18. The given expression can be simplified to $i^{2010-2008} + i^{2009-2008} = i^2 + i$. We have that $i = \sqrt{-1}$ and $i^2 = -1$, from which we find that our answer is $-1 + i$ or $i - 1$.

19. The probability that it will rain at least once is $1 -$ the probability that it won't rain at all. The probability that it won't rain at all is $(\frac{2}{5})^4$, or $\frac{16}{625}$. Therefore the probability that it will rain at least once is $\mathbf{\frac{609}{625}}$.

20. Factoring $4x^2 - 9y^2$, we have $(2x + 3y)(2x - 3y) = 15$. Since $2x + 3y = 3$, we have $2x - 3y = \mathbf{5}$.

21. By definition, an octahedron has 8 faces that are all equilateral triangles, 12 edges, and 6 vertices. Therefor, an octahedron has **6** vertices.

22. To find all the solutions to the equation, start from $x = 1$, $y = 2003$, and increase x by 1. When x reaches 288, y has to be negative to satisfy the equation, thus the maximum possible x is 287. Since x can be any integer from 1 to 287, there are **287** possible values of x.

23. First of all, $2^3 \cong 8 \cong -1$ $(mod\ 9)$. Then, $2^9 \cong -1^3 = -1$ $(mod\ 9)$, and $2^27 \cong -1^3 = -1$ $(mod\ 9)$ Obviously, -1 $(mod\ 9) = 8$ $(mod\ 9)$, so the answer is **8**.

24. If the first bead is green, then the last bead must be green also, so there are ${}_8C_4$ ways to order the remaining beads. If the fist bead is yellow, then the last bead must be yellow also, so there are ${}_8C_2$ ways to order the remaining beads. Therefore the answer is $\frac{{}_8C_4}{{}_8C_4 + {}_8C_2}$, which is $\frac{70}{70+28} = \mathbf{\frac{5}{7}}$.

25. The angle between the hour hand and 3 o'clock is $0.5 \times 32 = 16^\circ$. The angle between 3 o'clock and the minute hand is $13 \times 6 = 78^\circ$. Hence, the minute hand is $180 - (16 + 78) = 86^\circ$ behind. The minute hand moves 6° a minute, and the hour hand moves 0.5° a minute. Therefore, there is a difference of 5.5° per minute. $\frac{86}{5.5} = \mathbf{\frac{172}{11}}$.

26. The probability the Master of Debates wins all three is $0.9^3 = 0.729$, and the probability he wins two games is $0.9^2 \times 3 \times 0.1 = 0.243$. Therefore, the probability he wins the best of three is $0.729 + 0.243 = \mathbf{0.972}$.

27. Let the mass of a normal Pokeball be x grams. Label the boxes $1, 2, 3, \ldots, 10$. Now, take out one ball from box 1, two balls from box $2, \ldots, 10$ balls from box 10, and put all 55 balls on the digital scale. If all Pokeballs on the scale were normal, we would have $55x$ grams. Now, if the scale shows $55x - n$ grams for $1 \leq n \leq 10$, since each defective Pokeball weigh one gram less than normal ones, box n contains the defective Pokeballs. Hence, only one (**1**) weighing is needed.

28. Assume that $x \neq 0$. We notice that for real numbers a, b, $(a + b)^2 = a^2 + 2ab + b^2$. So, $a^2 + b^2 = (a + b)^2 - 2ab$. In the context of this problem, $x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2(x \times \frac{1}{x}) = 1^2 - 2 = \mathbf{-1}$, since $x + \frac{1}{x} = 1$.

29. Let a, b, c, $d =$ the ages of David Yang's children. Though $a \times b \times c \times d = 7$, 7 is a prime number, thus its only factors are 1 and 7. One of a, b, c, and d must be 7, and the rest must be 1, as a, b, c, and d are positive integers. $7 + 1 + 1 + 1 = \mathbf{10}$.

30. In the 10 by 10 square, the length is increased by 10% and the width is decreased by 10%. So the length is 11 and the width is 9, and therefore the area of the rectangle is 99. This area is 99% of the original area, and hence there is a change of **1**% in area.

31. When the stock price falls 10%, the resulting price is $(1 - 0.1) \times 600 = \$540$. When it rises by 10%, the resulting price is $(1 + 0.1) \times 540 = \$\mathbf{594}$.

32. The smallest sum occurs when the summands are as small as possible. The smallest possible squares are 1, 4, 9, and 16. Therefore the sum is $1 + 4 + 9 + 16 = \mathbf{30}$.

33. To count to one thousand, he would need to write one four-digit number ("1000"), nine hundred three-digit ones ("100" to "999"), ninety two-digit ones ("10" to "99"), and ten one-digit ones ("0" to "9"). That's a total of $4 \times 1 + 3 \times 900 + 2 \times 90 + 10 = 2894$ digits. Since on each day the boy writes 120 digits, 2894 of them would take 25 days ($\frac{\mathbf{2894}}{\mathbf{120}} \approx \mathbf{24.1167}$).

34. Ignoring the condition that Victoria's shirt and pants have to be of different colors, there are total of $5 \times 3 = 15$ different outfits she can make. However, for three of these outfits (both blue, black, brown), her shirt and pants are the same color. Hence, we subtract 3 from 15, which gives **12.**

35. Let $2.4818181818181 \cdots = x$. Then, since there are two digits in the repeating unit of 81, find $99x$, that is: $100x - x = 248.1818181818... - 2.481818181... = 245.7 = 99x$. Rewrite 245.7 as an improper fraction, or $\frac{2457}{10}$. Then, $x = \frac{2457}{990} = \frac{273}{110}$. The desired sum is therefore $273 + 110 = \mathbf{383}$.

36. It can be estimated that $32000011^2 \approx 32000000^2 = 1024000000000000$. (The number of digits is clearly at least **16**; to be sure that the answer is 16, we note that 10^8 is the smallest number whose square has 17 digits. The answer must therefore have less than 17 digits, since $32(10^6) < (10^8)$.

37. Let $y = \frac{1}{x}$. Then $y + 3 = 13y$, so $3 = 12y$, so $y = \frac{1}{4}$. Hence, $\frac{1}{x} = \frac{1}{4}$, so $x = \mathbf{4}$.

38. We know that any two lines cannot intersect more than once, and that there are $\binom{5}{2} = 10$ possible pairs of lines within 5 lines, so we cannot have more than 10 points of intersection. We show that 10 points of intersection is possible by drawing a regular pentagon and extending the edges to form a 5 pointed star shape, in which there are **10** points of intersection.

39.
$$\frac{59}{\sqrt{3}+\sqrt{5}+\sqrt{7}}$$
$$= \frac{59(\sqrt{3}+\sqrt{5}-\sqrt{7})}{(\sqrt{3}+\sqrt{5}+\sqrt{7})(\sqrt{3}+\sqrt{5}-\sqrt{7})}$$
$$= \frac{59(\sqrt{3}+\sqrt{5}-\sqrt{7})}{(\sqrt{3}+\sqrt{5})^2-(\sqrt{7})^2}$$
$$= \frac{59(\sqrt{3}+\sqrt{5}-\sqrt{7})}{2\sqrt{15}+1}$$
$$= \frac{59(\sqrt{3}+\sqrt{5}-\sqrt{7})(2\sqrt{15}-1)}{(2\sqrt{15}+1)(2\sqrt{15}-1)}$$
$$= \frac{59(\sqrt{3}+\sqrt{5}-\sqrt{7})(2\sqrt{15}-1)}{(2\sqrt{15})^2-1^2}$$
$$= \frac{59(\sqrt{3}+\sqrt{5}-\sqrt{7})(2\sqrt{15}-1)}{59}$$
$$= (\sqrt{3}+\sqrt{5}-\sqrt{7})(2\sqrt{15}-1)$$
$$= 6\sqrt{5}+10\sqrt{3}-2\sqrt{105}-\sqrt{3}-\sqrt{5}+\sqrt{7}$$
$$= 9\sqrt{3}+5\sqrt{5}+\sqrt{7}-2\sqrt{105}.$$
Hence, $a = 9$, $b = 5$, $c = 1$, and $d = 2$, so $a + b + c + d = \mathbf{17}$.

40. Let x be the side length of the square. The area of a square is $x \times x = x^2$. The perimeter of a square is $x+x+x+x = 4x$. Since the area of the square is 9 times more than the perimeter, we have that $x^2 = 9 \times (4x)$. Thus, $x^2 = 36x$. Dividing both sides by x yields $x = \mathbf{36}$.

41. Because the polynomial passes through $(0, 0)$, $(1, 0)$, $(2, 0)$, and $(3, 0)$, we know that it has roots at $x = 0$, 1, 2, and 3. That means that the polynomial can be factored as $Ax(x-1)(x-2)(x-3)$. Then the value of the polynomial at $x = 4$ is $A \times 4 \times 3 \times 2 \times 1$, which is $24A$. Since we know that the value at $x = 4$ is 1, A must be $\frac{1}{24}$. Hence, the value of polynomial when $x = 5$ is $\frac{1}{24} \times 5 \times 4 \times 3 \times 2$, or **5**.

42. We want to find the largest power of 5 that fully divides 2010!. There are 402 numbers that are divisible by 5, 80 that are divisible by 25, 16 divisible by 125, and 3 divisible by 625. Therefore, the answer is $402+80+16+3 =$ **501**.

43. Consider the triangle ABC, and consider the center of the triangle O. Draw $\overline{AO}$. Let M be the midpoint of $\overline{AB}$. Draw $\overline{OM}$. $\overline{OM} \perp \overline{AB}$. $\triangle AMO$ is a right triangle with angles $30°, 60°$, and $90°$. $\overline{AB} = 4$, so $\overline{AM} = 2$. Let $x =$ the opposite side of the $30°$ angle. As the opposite side of the $60°$ angle, $2 = x\sqrt{3}$, so $x = \frac{2\sqrt{3}}{3}$. The radius of the circle is $\overline{OA}$. As the opposite side of the $90°$ angle, $\overline{OA} = 2x$, thus $\overline{OA} = \mathbf{\frac{4\sqrt{3}}{3}}$.

44. Let R be the radius of $\mathcal{C}_1$, and r be the radius of $\mathcal{C}_2$. The area we are looking for is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$. Drawing a triangle with one of the points where the chord meets $\mathcal{C}_1$, the point where the chord is tangent to $\mathcal{C}_2$ and the center of the circles, we can easily see that we have a right triangle with half of the cord as a side. Applying the Pythagorean Theorem yields $r^2 + 6^2 = R^2$ or $R^2 - r^2 = 6^2 = 36$, and from that, we have $\pi(R^2 - r^2) = \mathbf{36}\pi$.

45. The sum should have been $50 \times 9 \div 2 = 225$. We have $225 + \overline{2a2b} - \overline{2a} - \overline{2b} = 2601$, which gives $99a = 396$ upon simplifying. Hence, $a = 4$. Noting that $b + 1$, we have $b = 5$. Hence, $a + b = \mathbf{9}$.

46. We write $2n^2 - 16$ as $2(n^2 - 8)$. $2(n^2 - 8)$ is not prime unless $n^2 - 8 = 1$ since 2 divides $2n^2 - 16$. Hence, $n^2 - 8 = 1$, so $n = \mathbf{3}$ or $n = \mathbf{-3}$.

47. Raising numbers to the 111th power preserves order, so it suffices to list $\sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \sqrt[5]{5}$ in an increasing order. Clearly $\sqrt{2} = \sqrt[4]{4}$. Since $8 = (\sqrt{2})^6 < (\sqrt[3]{3})^6 = 9$, we have $\sqrt{2} = \sqrt[4]{4} < \sqrt[3]{3}$. Also, since $25 = (\sqrt[5]{5})^{10} < (\sqrt{2})^{10} = 32$, we have $\sqrt[5]{5} < \sqrt{2} = \sqrt[4]{4} < \sqrt[3]{3}$. Hence, x and y are $\sqrt{2}$ or $\sqrt[4]{4}$ or vice versa. Either way, $\log_x y = \mathbf{1}$.

48. Note that $\frac{|x-y|}{2} \geq 0, \sqrt{2y+z} \geq 0$, and $z^2 - z + \frac{1}{4} = (z - \frac{1}{2})^2 \geq 0$. Since their sum is equal to 0, we find that equality must hold in each of the inequalities, so $z = \frac{1}{2}$, $x = y$, and $2y + z = 0$. Hence, $z = \frac{1}{2}$, $y = -\frac{1}{4}$, $x = -\frac{1}{4}$, so $(y+z)^x = \left(\frac{1}{4}\right)^{-\frac{1}{4}} = \sqrt{\mathbf{2}}$.

49. The prime factorization of 16000 is $2^7 \times 5^3$. To get 10 terminating zeros, we need 10 pairs of 2's and 5's. Since we already have seven 2's and three 5's, we only need three more twos and seven more fives to get 10 terminating zeros. Thus, the smallest integer that 16,000 can be multiplied to get 10 terminating zeros is $2^3 \times 5^7 =$ **625000.**

50. The numbers satisfying the given conditions take the form of $5\overline{aba}5$. There are 10 possibilities each for a and b, so there are a total of **100** combinations.