1. Recall that $10^n = \underbrace{100\ldots00}_{n \text{ zeros}}$, hence $10^{10}+10^8+10^6+10^4+10^2+10^0 = \boxed{10101010101}$.

2. There are 16 words in the sentence, and exactly 5 of them have four letters, as shown: "**What** is the probability **that** a randomly chosen **word** of **this** sentence has exactly **four** letters?"

   Therefore, the desired probability is simply $\boxed{\dfrac{5}{16}}$.

3. Since every flower has 7 petals, and Alex has 1001 petals, there are $\frac{1001}{7} = \boxed{143}$ flowers.

4. Since 30 minutes is $\frac{1}{2}$ of an hour, 36 apples will take $36 \cdot \frac{1}{2} = \boxed{18}$ hours to clone.

5. Applying the definition, we have $2\#0 = 2\cdot 0 - 2 - 3 = -5$ and $1\#4 = 1\cdot 4 - 1 - 3 = 0$. Thus,

$$(2\#0)\#(1\#4) = 5\#0 = (-5)\cdot 0 - (-5) - 3 = \boxed{2}.$$

6. After n days, his number of friends will increase by $5n$. Hence, after 2013 days, he has $4 + 2013(5) = \boxed{10069}$ friends.

7. Let the number in question be x. Then we know

$$21 + \frac{1}{4}x = \frac{3}{5}x \implies 21 = \left(\frac{3}{5} - \frac{1}{4}\right)x$$

$$\implies 21 = \left(\frac{12}{20} - \frac{5}{20}\right)x = \left(\frac{7}{20}\right)x$$

$$\implies x = 21 \cdot \frac{20}{7} = \boxed{60}.$$

   Verifying, $\dfrac{1}{4}$ of 60 is 15, and $\dfrac{3}{5}$ of 60 is 36. As $21+15$ is indeed equal to 36, this check is successful.

8. We can prime factorize $12 = 2^2 \cdot 3$ and $18 = 2 \cdot 3^2$. Hence, since the greatest common divisor takes the smallest exponent of each prime, we have $\gcd(12, 18) = 2 \cdot 3 = 6$. Similarly, since the least common multiple takes the largest exponent of each prime, we have $\operatorname{lcm}(12, 18) = 2^2 \cdot 3^2 = 36$. Thus our answer is $6 + 36 = \boxed{42}$.

   Alternatively, we can utilize the Euclidean Algorithm to find $\gcd(12, 18) = \gcd(12, 6) = \gcd(6, 6) = 6$, then use $\operatorname{lcm}(x, y) = \dfrac{xy}{\gcd(x, y)}$ to find $\operatorname{lcm}(12, 18) = \dfrac{12 \cdot 18}{6} = 36$. Our answer is then the same as before.

9. Since 2 darps is equal to 4 derps, 6 darps is equal to 12 derps. Similarly, since 3 derps is equal to 5 dirps, 12 derps is equal to 20 dirps. Hence, 6 darps is equivalent to $\boxed{20}$ dirps.

10. The 7 smallest prime numbers are $2, 3, 5, 7, 11, 13$, and 17, the sum of which is $\boxed{58}$.

11. Suppose there are g girls in the room. Then there are $2g$ teachers and $g+6$ boys in the room, for a total of $4g+6 = 38$ people. Thus $g = 8$, and there are $8 + (8 + 6) = \boxed{22}$ children.

12. Since the length of Alex's rectangle and the width of Alex's rectangle are both 3 times the length and width of Kelvin's rectangle, the area of Alex's rectangle is $3 \cdot 3 = 9$ times the area of Kelvin's rectangle. Hence the area of Alex's rectangle is $9 \cdot 12 = \boxed{108}$.

13. We have $P(1) = (1 + 1)(1 + 2)(1 + 3)\ldots(1 + 2013)(1 + 2014) = 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 2014 \cdot 2015$ and $P(0) = (0+1)(0+2)(0+3)\ldots(0+2013)(0+2014) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2013 \cdot 2014$. Thus,

$$\frac{P(1)}{P(0)} = \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \ldots \cdot \cancel{2014} \cdot 2015}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \ldots \cdot \cancel{2013} \cdot \cancel{2014}} = \frac{2015}{1} = \boxed{2015}.$$

Alternatively, the exact same cancellation shows that $\dfrac{P(x)}{P(x-1)} = \dfrac{x + 2014}{x}$, hence $\dfrac{P(1)}{P(0)} = \dfrac{1 + 2014}{1} = 2015$.

14. Suppose the side length of the cube is s. Then the area of each face of the cube, being a square, has area s^2. Since there are 6 such faces, the surface area of the cube is $6s^2$. As such we have $6s^2 = 294 \implies s^2 = 49 \implies s = \pm 7$. Clearly $s = -7$ does not make sense as a side length, leaving the solution $s = \boxed{7}$.

15. Let his final exam score be x. Then

$$\frac{91 + 89 + 88 + 94 + 87 + 85 + x}{7} = x$$
$$\implies \frac{534 + x}{7} = x$$
$$\implies 534 + x = 7x \implies 534 = 6x$$
$$\implies x = \boxed{89}$$

16. Let there be n nickels and d dimes. Since there are 70 coins in total, the total value of which is 555 cents, we have the system of equations:

$$n + d = 70 \implies 5n + 5d = 350$$

$$5n + 10d = 555$$

Subtracting the first equation from the second gives $5d = 205 \implies d = 41$, implying that $n = 29$. Checking, there are indeed 70 coins, and the nickels are worth $\$0.05 \cdot 29 = \$1.45$ and the dimes are worth $\$0.10 \cdot 41 = \$4.10$, which does indeed sum up to \$5.55. Therefore our answer is $d - n = 41 - 29 = \boxed{12}$.

17. Let the integers be $x - 5, x - 4, \ldots, x + 4, x + 5$. Then we have $11x = 1331 \implies x = 121$, making the largest of them $x + 5 = 121 + 5 = \boxed{126}$.

18. Cross-multiplying, the equation is equivalent to $(x^2 + 1)(x + 1) = (x^2 - 1)(x-1)$. Expanding gives $x^3+x^2+x+1 = x^3-x^2-x+1 \implies 2(x^2+x) = 0 \implies x^2 + x = 0 \implies x(x + 1) = 0$. Thus $x = 0$ or $x = -1$. However, if $x = -1$, then the fraction $\dfrac{x^2 - 1}{x + 1}$ will have denominator zero, meaning that the only valid solution is $x = \boxed{0}$.

19. Though the top floor is labeled "88", we have skipped all the floors containing a 4. That includes the floors $4, 14, \ldots, 84$ and $40, 41, \ldots, 49$, of which there are 18 (there are 9 in the first list and 10 in the second, but both lists contain 44). Hence there are actually only $88 - 18 = \boxed{70}$ floors in this building.

20. The radius of the garden and the path together is $6 + 3 = 9$, hence the area is $9^2\pi = 81\pi$. The area of the garden is $6^2\pi = 36\pi$, so the area of the path is $81\pi - 36\pi = \boxed{45\pi}$.

21. Let a and b be the integers, and without loss of generality assume that $a \geq b$ (if $a < b$, we can simply switch the two). Since $8 = a + b \leq 2a$, we know that a is at least 4. Furthermore, since $a^2 \leq a^2 + b^2 = 34$, we know that a is less than 6. Therefore we only need to check $a = 4$ and $a = 5$, the latter of which leads to the solution $a = 5, b = 3$. Therefore, $ab = 5 \cdot 3 = \boxed{15}$.

22. To walk 42 feet away from his house at the rate of 3 feet per second, Young Guy needs 14 seconds. To run back at the rate of 7 feet per second, Young Guy needs only 6 seconds. Hence he covered 84 feet in 20 seconds, for a total average speed of $\boxed{4.2}$ feet per second.

23. Note that the surface area varies with the square of the side length, and the volume varies with the cube of the side length. In other words, multiplying the side length by x is the same as multiplying the surface area by x^2 and the volume by x^3. In this case, we are adding 20% to the side length, equivalent to multiplying by 1.2. Therefore, the surface area of the cube is increased by a factor of 1.44, or 44%, and the volume of the cube is increased by a factor of 1.728, or 72.8%. Thus $x = 44$ and $y = 72.8$, meaning $5(y - x) = 5(72.8 - 44) = \boxed{144}$.

24. On his first run he breaks through $\frac{3}{2}$ inches of wood, on his second he breaks through $\frac{9}{8}$, on his third he breaks through $\frac{27}{32}$ (which is still insufficient), and on his fourth he breaks through $\frac{81}{128}$. The sum of these is $\frac{525}{128}$, which is greater than 4, so he must run into the wall $\boxed{4}$ times.

25. There are two cases to consider. The first is when Alex wins on his first turn, and the second is when Alex wins on his second turn. Notice that Alex will never get a third turn, as by then all the spots will necessarily be winning. The first case necessitates Kelvin not rolling an even number on his first turn, which occurs with a probability of $\dfrac{1}{2}$, and Alex rolling either an even number or the number Kelvin rolled, which occurs with a probability of $\dfrac{2}{3}$. Hence the probability of this case occurring is $\dfrac{1}{3}$. The second case requires Kelvin rolling an odd number, followed by Alex not winning, followed by Kelvin rolling the only number remaining that isn't winning. Then all 6 numbers will be winning, and Alex will win. The probability of this case is $\dfrac{1}{2} \cdot \dfrac{1}{3} \cdot \dfrac{1}{6} = \dfrac{1}{36}$, making the total probability $\dfrac{1}{3} + \dfrac{1}{36} = \boxed{\dfrac{13}{36}}$.

26. Notice that $4^x - 1 = (2^x)^2 - 1 = (2^x - 1)(2^x + 1)$, meaning that $4^x - 1$ can be prime only if $2^x - 1 = 1$ or $2^x + 1 = 1$. The latter is impossible as there is no positive integer x such that $2^x = 0$, while the former leads to $2^x = 2 \implies x = 1$. This is the only such value, and thus our answer is $\boxed{1}$.

    Alternatively, notice that 4^x always leaves a remainder of 1 upon division by 3, and thus $4^x - 1$ is always divisible by 3. As such, the only way $4^x - 1$ can be prime is if $4^x - 1 = 3 \implies 4^x = 4 \implies x = 1$, and the rest is as before.

27. Suppose there are y problems written thus far. Then

$$\frac{x+x}{y+x} = \frac{1}{2}$$
$$\implies \frac{2x}{x+y} = \frac{1}{2}$$
$$\implies 4x = x + y \implies y = 3x$$

    and therefore Alex has written $\frac{x}{3x} = \boxed{\dfrac{1}{3}}$ of the current submissions.

28. Let the side length of the cube be a and the side length of the square be b. Then $a^3 = b^2$. Suppose p is a prime dividing a. Since $p^3 \mid a^3$, then $p^2 \mid b$. But then $p^4 \mid b^2$, so $p^2 \mid a$ and thus $p^3 \mid b$. Hence, a is a perfect square and b is a perfect cube. The smallest perfect square greater than 1 is 4, so $a = 4$ and the volume is $4^3 = \boxed{64}$.

29. Since the triangle's altitudes are all equal, the triangle must be equilateral. Suppose the side length of this triangle is $2s$. Then, by the Pythagorean theorem, $s^2 + 6^2 = (2s)^2 \implies 36 = 3s^2 \implies 12 = s^2$. The area of the triangle is thus $\frac{s^2\sqrt{3}}{4} = \frac{12\sqrt{3}}{4} = \boxed{3\sqrt{3}}$.

30. Let X and Y be the altitudes from E to AB and G to CD respectively. Then, by the Pythagorean theorem, $EX = GY = \frac{\sqrt{3}}{2}$. Hence $EG = \sqrt{3} + 1$. Therefore, the area of $EFGH$ is

$$\frac{EG^2}{2} = \frac{(\sqrt{3}+1)^2}{2} = \boxed{2+\sqrt{3}}.$$

31. To get at least 21 points on the USAMO, Alex will need to expend at least 210 units of effort. To get at least 200 points on the AMC and AIME combined, Alex can expend 102 units of effort by focusing on the AMC alone, or 96+7=103 units of effort by getting 192 points on the AMC and 10 on the AIME. Any lower amount of effort on the AMC is clearly counterproductive, as he gets a higher rate of points to effort on the AMC than on the AIME. Hence the minimum possible amount of effort he can expand is $210 + 102 = \boxed{312}$.

32. Let the number be $\overline{abc}$. If any of a, b, c are zero, then the product of the digits would be zero, implying that the sum of the digits is 0 - impossible. Hence a, b, c are all positive digits. WLOG assume that $a \le b \le c$ - we'll account for the permutations later. Thus we have $abc = a + b + c \le 3c \implies ab \le 3$, meaning we need check only $(a, b) = (1, 1)$, $(a, b) = (1, 2)$, and $(a, b) = (1, 3)$. The former case gives us $c = 2 + c$, contradiction, the second gives us $2c = 3 + c \implies c = 3$, and the third gives us $3c = 4 + c \implies c = 2$, a contradiction as we assumed $b \le c$. Therefore the only possible numbers are the permutations of $\overline{123}$, of which there are $3! = \boxed{6}$.

33. We first count the number of arrangements such that the girls *are* next to each other. Then this is equivalent to arranging 4 positions, one for each of the three boys and one for the adjacent girls, and arranging the 2 girls in their given position. There are $4! \cdot 2! = 48$ ways of doing this, and there are $5! = 120$ possible arrangements ignoring the condition. Hence, there are a total of $120 - 48 = \boxed{72}$ arrangements without the girls next to each other.

34. Notice that $x^2 = 2^{64} \implies x = 2^{64\cdot\frac{1}{2}} = 2^{32}$. Then $x^x = (2^{32})^{2^{32}} = 2^{32\cdot 2^{32}} = 2^{2^{37}} = 2^y$, making $y = 2^{37}$. Finally, $y = 2^z \implies 2^{37} = 2^z \implies z = \boxed{37}$.

35. Since the triangle is isosceles, some two of its side lengths are equal. Hence either $x - 4 = 2x - 9 \implies x = 5$, $2x - 9 = 3x - 15 \implies x = 6$, or $x - 4 = 3x - 15 \implies x = \frac{11}{2}$. However, if $x = 5$, the side lengths of the

triangle are 1, 1, and 0 - clearly not a triangle. Hence the only two valid values of x are 6 and $\frac{11}{2}$, the sum of which is $\boxed{\frac{23}{2}}$.

36. If $x \leq 4$, then the median of the numbers is 4 and hence $\frac{24+x}{5} = 4 \implies x = -4$. If $4 < x < 7$, then the median of the numbers is x and hence $\frac{24+x}{5} = x \implies x = 6$. If $x \geq 7$, then the median of the numbers is 7 and hence $\frac{24+x}{5} = 7 \implies x = 11$. The sum of these three values is thus $(-4) + 6 + 11 = \boxed{13}$.

37. Let O be the center of the circle, and A, B be two adjacent vertices of the polygon. Since the area of the circle is 16π, its radius is 4, and hence $OA = OB = AB = 4$. Therefore OAB is equilateral, and $\angle AOB = 60°$. There are thus $\frac{360°}{60°} = \boxed{6}$ sides to the polygon.

38. After $3n$ stops, $\frac{3n(3n+1)}{2}$ passengers have gotten on the bus and $2n$ have left, for a total of $\frac{9n^2-n)}{2}$ passengers on the bus. Hence, after 9 stops there are 39 passengers on the bus. After the 10th stop there are thus 49 passengers, and after the $\boxed{11}$th the bus is full.

39. Clearly this computation is infeasible to do directly, so we need to find a more clever method. Let $x = 102$ and $y = 2$. Then

$$102^4 - 8 \cdot 102^3 + 24 \cdot 102^2 - 32 \cdot 102 + 16 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4,$$

which is precisely the expansion of $(x - y)^4$! Therefore our answer is $(102 - 2)^4 = 100^4 = \boxed{100000000}$.

40. To have exactly seven factors, the number must be of the form p^6 for some prime p. Minimizing this is equivalent to minimizing p, so we want p to be the smallest prime - 2. Our desired number is thus $2^6 = \boxed{64}$, which indeed has the seven factors 1, 2, 4, 8, 16, 32, and 64.

41. There are three ways this is possible. Either $x + 4 = 1$, in which case $(x+4)^{3x+1} = 1$ as $1^n = 1$ for any n, or $3x + 1 = 0, x + 4 \neq 0$ as $n^0 = 1$ for any $n \neq 0$, or $x + 4 = -1$ and $3x + 1$ is even, as $(-1)^n = 1$ for any even n. These lead to the three valid solutions $x = -3, x = -\frac{1}{3}, x = -5$, the product of which is $\boxed{-5}$.

42. Let O_1, O_2, and O_3 be the centers of the three unit circles, and ABC be the desired equilateral triangle. Dropping altitudes from O_1 and O_2 to $\overline{AB}$, meeting the line at C and D respectively, gives $AB = AC + CD + DB$. Since $O_1C = 1$ and $\angle ABO_2 = \angle BAO_1 = 30°$, we have $AC = DB = \sqrt{3}$. Furthermore, $CD = O_1O_2 = 2$. Thus $AB = 2 + 2\sqrt{3}$, making the area of ABC equal to $\frac{s^2\sqrt{3}}{4} = (\sqrt{3} + 1)^2\sqrt{3} = \boxed{6 + 4\sqrt{3}}$.

43. There are 10 pages with 1-digit numbers (0 through 9), and 90 pages with 2-digit numbers (10 through 99). This accounts for $1\cdot10+2\cdot90 = 190$ digits, leaving $2014-190 = 1824$ digits left to be written. At 3 digits apiece, this accounts for 608 more pages. Thus there are a total of $10+90+608 = \boxed{708}$ pages.

44. Notice that $a_0 = 1, a_1 = \sqrt{3\cdot1+1} = 2, a_2 = \sqrt{4\cdot2+1} = 3$, and so on. Thus I claim that $a_n = n+1$. Since $n+1 = \sqrt{(n+2)\cdot n+1} = \sqrt{(n+2)\cdot a_{n-1}+1}$, this is true. Therefore $a_{2014} = 2014+1 = \boxed{2015}$.

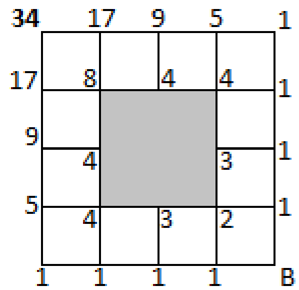
45. Let the square be $ABCD$ and suppose that they stand at points A and C. Then the desired region is the intersection of the unit circles centered at A and C. Notice that the quarter-circle ABD has area $\frac{\pi}{4}$, while *triangle* ABD has area $\frac{1}{2}$. Hence the area formed by $\overline{BD}$ and *arc* BD is $\frac{\pi}{4}-\frac{1}{2}$. Twice this area, the desired region, is $\boxed{\frac{\pi}{2}-1} = \boxed{\frac{\pi-2}{2}}$.

46. If 15 is one of the legs, then we have $15^2+x^2 = (2x)^2 \implies 15^2 = 3x^2 \implies x = \frac{15}{\sqrt{3}} = 5\sqrt{3}$ where x is the length of the other leg. If 15 is the hypotenuse, then we have $x^2+(2x)^2 = 15^2 \implies 5x^2 = 15^2 \implies x = \frac{15}{\sqrt{5}} = 3\sqrt{5}$, where x is the length of the shorter leg. The product of these is $\boxed{15\sqrt{15}}$.

47. Since the sum of the numbers from 1 to k is $\frac{k(k+1)}{2}$, we have $24 \mid \frac{k(k+1)}{2}$. As k and $k+1$ are relatively prime, and $8 \mid \frac{k(k+1)}{2}$, either $16 \mid k$ or $16 \mid k+1$. Hence the smallest possible value is $k = \boxed{15}$, which is indeed valid as $1+2+3+\ldots+15 = 120$ - a multiple of 24.

48. It is not difficult to guess-and-check the solution, but we will investigate a more general method. From $ab+cd = a+b+c+d$ we get $(ab-a-b)+(cd-c-d) = 0 \implies (ab-a-b+1)+(cd-c-d+1) = 2$. Therefore, $(a-1)(b-1)+(c-1)(d-1) = 2$. If none of a,b,c,d are 1, then we would necessarily have $(a-1)(b-1) = 1$ and $(c-1)(d-1) = 1$ as a,b,c,d are positive integers. This would give us $a = b = c = d = 2$, which does not satisfy either equation. Thus we can assume $a = 1$. This implies that $(c-1)(d-1) = 2$, implying that $c = 3$ and $d = 2$ or the reverse. Finally, $a+b+c+d = 11$ gives us $1+b+3+2 = 11 \implies b = 5$. Finally, $abcd = 1\cdot5\cdot3\cdot2 = \boxed{30}$.

49. Mark each point with the number of paths from it to B. Then, since our only options are to move one unit to the right or one unit down, each number is the sum of the number below it (if any) and the number to the right of it (if any). Clearly, any point on the right or bottom edges has only one possible path - all moves down or all moves right, respectively. The picture shows this process, leading to the answer of $\boxed{34}$.



Alternatively, notice that the only "illegal" paths are the ones passing through the center point, since that is the only one in the middle hole. Hence any combination of 4 moves to the right and 4 moves down, of which there are $\binom{8}{4} = 70$ arrangements, will result in a valid path - unless it passes through the center points, which occurs in $\binom{4}{2}^2 = 36$ of these paths. Hence the answer is $70 - 36 = \boxed{34}$ as before.

50. Since $\angle ABC$ is right, B lies on the circle with diameter AC, and therefore $MA = MB = MC = 13$. Thus

$$r_1 \cdot \frac{13+13+10}{2} = [ABM] = \frac{1}{2}[ABC] = 60$$

and

$$r_2 \cdot \frac{13+13+24}{2} = [ACM] = \frac{1}{2}[ABC] = 60$$

Hence $r_1 r_2 = \frac{120^2}{36 \cdot 50} = \boxed{8}$.