2015 Joe Holbrook Memorial Math Competition
5th Grade Exam Solutions

The Bergen County Academies Math Team

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1. This is straightforward addition. $11 + 10 + 9 + 7 + 5 + 4 + 2 + 1 + 1 = \boxed{50}$.

2. The number of water bottles he drinks in a day is the sum of the numbers of water bottles he drinks at each meal. $2 + 3 + 5 = \boxed{10}$.

3. Sixteen years ago, Sunshine was four years old, while Moonbeam was not yet born, so $\boxed{\text{Moonbeam}}$ was born second.

4. (Thomas's weight) + (the book's weight) = 350, so

   (Thomas's weight) = 350 - (the book's weight) = $350 - 5 = \boxed{345}$.

5. There are 11 fish, each of which eats 19 flakes, so the total amount that Nemo needs can be computed by $11 \cdot 19 = \boxed{209}$.

6. By order of operations, we multiply and divide before adding and subtracting. $14+4\cdot\frac{15}{3}-9 = 14 + 4 \cdot 5 - 9 = 14 + 20 - 9 = \boxed{25}$.

7. All multiples of 5 end in a 0 or a 5. This means that all numbers ending in a 1 or a 6 have a remainder of $\boxed{1}$.

8. The Sun sets every evening, which comes after noon. Thus between noon Saturday and noon Thursday, Saturday, Sunday, Monday, Tuesday, and Wednesday have evenings; this makes $\boxed{5}$ days.

9. The first term of the sequence is 1, and each subsequent term is 3 more than the previous one, so the second term is $1+3 = 4$, the third term is $1+3+3 = 7$, and so on. Extending this, the nth term is $1+3(n-1)$, so the 100th term is $1+3(100-1) = 1+3(99) = 1+297 = \boxed{298}$

10. $3(3(3+2)+2(3+2))+2(3(3+2)+2(3+2))$

    $= (3+2)(3(3+2)+2(3+2))$

    $= (3+2)((3+2)(3+2)) = 5^3 = \boxed{125}$.

11. There are going to be 6 bonsai-to-bonsai gaps (as gap count = $\frac{\text{total length}}{\text{gap length}}$). Each gap has 2 endpoints, so there are going to be $\boxed{7}$ trees total.

12. A pyramid is the shape formed by a polygonal base and the line segments from each of the vertices of the base to another point in a different plane, called the *vertex* of the pyramid. If the base is octagonal, then there are 8 edges on the octagon and 8 edges from each of the vertices, for a total of 16; there is the octoganal face, with 8 triangular faces formed by the vertex-edges and the edges of the octagon, for a total of 9. Thus the answer is $16 + 9 = \boxed{25}$.

13. Suppose Kelvin's roll is written as an ordered pair, with the first number coming from the first die. Then, the rolls that obtain a 5 are $(1,4)$, $(2,3)$, $(2,3)$ (duplicated because 2 appears twice on the first one), $(4,1)$. There are 4 desired outcomes out of $6 \times 6 = 36$ total possible outcomes, for a final answer of $\frac{4}{36}$, or $\boxed{\frac{1}{9}}$.

14. Recognizing that each number is one more than the one before it, we can group the terms into pairs:

    $0+(-1+2)+(-3+4)+\cdots+(-99+100) = 0+1+1+\cdots+1$. Each positive even number ends a pair, and there are 50 positive even integers less than 100, so the final sum is $\boxed{50}$

15. If there are 2015 consecutive integers, then they must be $n-1007, n-1006, \cdots n-1, n, n+1, \cdots n+1006, n+1007$. When computing the sum, each $n-k$ is canceled out by $n+k$, so the total sum is $2015n$ and the mean is n, which is also the median of the list. The mean is 2, so $\boxed{2}$ is the median as well.

16. Three-digit numbers are greater or equal to 100 and less than 1000. The least multiple of 17 greater than 100 is $17 \cdot 6 = 102$, and the greatest less than 1000 is $7 \cdot 58 = 986$. This means there are $58 - 6 + 1 = \boxed{53}$ multiples.

17. If Sung Hyup was 15 years old 4 years ago, he is now 19 years old. Sunny is 3 years younger than him, so he is currently 16. In two years he will be $16 + 2 = \boxed{18}$

18. The probe spent every month in the years 2007-2014 in space, which is $12 \cdot 8 = 96$ months. In addition, it spends every month except January in 2006, and every month before July in 2015. Thus the total number of calendar months spent in space is $96 + 11 + 6 = \boxed{113}$.

19. There are 2 hours and 29 minutes from 5:00 to 7:29. Each hour is 60 minutes long, so this is $2(60) + 29 = 149$ minutes. Each episode is 20 minutes long, so the number of full episodes that cam be watched is the integer part of $\frac{149}{20}$, which is $\boxed{7}$.

20. First, we compute what Matt pays. His discount is $10\% \times \$20 = 0.1 \times \$20 = \$2$, so he pays \$18 total. Tanny pays $\$25 = \$20 + \$5$, so his 15% discount applies only to \$20: $15\% \times \$20 = 0.15 \times \$20 = \$3$ is his discount, so he pays in total \$22. Thus, the answer is $\boxed{\text{Tanny, \$22}}$.

21. Let's track their progress using coordinates, setting $x+$ to be forward movement and $y+$ to be upward movement. Their path is as follows, in relative coordinates: (0,0), (0,10), (0.5,10), (0.5,2), (25.5,2), (25.5,11), (27,11), (27,1), (37,1). Their x-path has length 37. Their y-path is the sum of the absolute values of the individual y-differences: $10+8+9+10 = 37$. Their total path has length $37 + 37 = \boxed{74}$.

22. To get the least product, we want to end with a big, negative number. To achieve this, we pick two big factors with opposite signs; $-7 \cdot 3 = \boxed{-21}$.

23. Alex catches up by $\frac{1}{2}$ of a foot every second. Since he needs to catch up on what is initially a 20 foot lead, he will take $\boxed{40}$ seconds to do this.

24. Simplifying the fraction and grouping some terms, we see that $(23+46\times 23+\frac{46}{2}) - 48\times 23 = (48 \times 23) - 48 \times 23 = \boxed{0}$.

25. A square has four equal sides, so its perimeter is four times a side length, i.e. $4 \cdot 18 = 72$. A hexagon has six sides, so each side is $\frac{1}{6}$ of the perimeter, which is same as the square's, so the side length is $\frac{72}{6} = \boxed{12}$.

26. Max's share is $1 - \frac{2}{5} - \frac{1}{10} - \frac{1}{4} = \frac{20-8-2-5}{20} = \frac{1}{4}$. Jen eats $\frac{2}{5} \times 20 = 8$ ounces, and Max eats $\frac{1}{4} \times 20 = 5$ ounces. Thus, the answer is $8 - 5 = \boxed{3}$ ounces.

27. Note that $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$ with last digit 6, and $2^5 = 32$ with last digit 2. Thus the lsat digits follow the pattern $2, 4, 8, 6, \ldots$. So we see that 2^{244} has last digit 6, and 2^{245} has last digit $\boxed{2}$.

28. Notice that this is asking for the LCM of 4, 5, and 6. First take 4 and 6. It is easy to see that the LCM is 12. Now, consider 5 and 12. Since they are relatively prime, their LCM is simply their product, which is $\boxed{60}$.

29. There are 10 dominoes with two of the same digit, one for each number from 0-9. There are then $\binom{10}{2}$ ways to choose the dominoes with two different digit. Thus there are $10 + \binom{10}{2} = 10 + 45 = \boxed{55}$ different dominoes.

30. Let x be the favorite number. Rewriting the question as a mathematical equation, we have that $3(x+9) = p^4$, where p is the smallest odd prime number; this is 3. So, $3(x+9) = 3^4 = 81 \implies 3x + 27 = 81 \implies 3x = 54 \implies x = 18$. We are looking for half of x, which is $\boxed{9}$.

31. Let's relabel the letters slightly: $\text{JHM}_1\text{M}_2\text{C}$. Since they are all different, there are $5! = 120$ arrangements. But in our original word, the M's were the same. Since in a given arrangement, there are $2! = 2$ ways to arrange them, we divide out by this factor of overcounting, to get $\boxed{60}$ arrangements.

32. $a_n = a_{n-1} + a_{n-2} + \cdots + a_0 = (a_{n-2} + \cdots + a_0) + (a_{n-2} + \cdots + a_0) = 2a_n - 1$ for $n > 3$. $a_3 = 3$, so $a_8 = 2^{8-3}a_3 = \boxed{96}$.

33. The digit in the seventh position is always removed, and there will always be a digit in the seventh position until there are less than seven digits total, i.e. when there are six; the first six digits are never touched, so the final number will be $\boxed{123456}$.

34. Think about the large cube as a $3 \times 3 \times 3$ cube of smaller unit cubes. Then, we have simply removed one cube from each face, plus the center cube–a total of 7 cubes. Thus, $27 - 7 = \boxed{20}$ is the remaining volume.

35. Switching digits will not reduce an integer if its first digit is greater than or equal to its second. Because of this the only two digit numbers that satisfy the condition with the first digit n are $nn, nn-1, \cdots, n0, n+1$ in total. Since a two digit number can only start with a number from 1 to 9, this means that there are (1 + 1) + (2 + 1) + ... + (9 + 1) = 54 valid two digit integers. And since there are 90 two digit integers, $\frac{3}{5}$ of them work.

36. Suppose the first point is fixed. The second point could be either 60° clockwise or counterclockwise from it, so the total region of the circumference is 120° degrees. $\frac{120°}{360°} = \boxed{\frac{1}{3}}$.

37. If he starts with x pencils on day n, then at the *beginning* of the $n^{th}, n-1^{st}, n-2^{nd}, n-3^{rd}, n-4^{th}, n-5^{th}$, and $n-6^{th}$ days are $x, x-1, x-2, x-3, x-4, x-1$, respectively. Thus at the beginning of a five-day cycle he has one less pencil than at the beginning of the previous one.

   $x - 4$ should never be nonpositive, i.e. $x - 4 \geq 1$, so he must begin the 21^{st} day with 4 pencils. Going backwards, on the 16^{th} he started with 5, on the 11^{th} with 6, the 6^{th} with 7 and the 1^{st} with $\boxed{8}$.

38. Alex travels the first leg of his trip in $\frac{300}{60} = 5$ hours, the second in $\frac{90}{90} = 1$ hour, and the third in $\frac{360}{40} = 9$ hours. The total length of time is $5 + 1 + 9 = 15$ hours. The total distance is $300 + 90 + 360 = 750$ hours. Thus, his average speed is $\frac{750}{15} = \boxed{50}$ miles per hour.

39. We will approximate these relative to each other. We note that

$$3^7 = 2187 > 2015. \text{ So } a = 3^{2015} > 3^{21} = (3^7)^3 > 2015^3 = b$$

We also note that $3^6 = 729 < 1009$.

$$c = 1009^{1009} > 1009^{336} > 729^{336} = (3^6)^{336} = 3^{2016} > 3^{2015} = a.$$

Thus, $\boxed{c > a > b}$.

40. We are given that $\frac{a+b+c}{3} = 46$. We want to find $\frac{(a-7)+(b+4)+(c+6)}{3} = \frac{a+b+c+3}{3} = \frac{a+b+c}{3} + 1 = \boxed{47}$. (It turns out that the sequence information was superfluous.)

41. This question is solved by plotting each function and noting all intersection points. $y = \frac{1}{x}$ intersects with $y = 1$ at $(1,1)$, with $y = x$ at $(1,1)$ and $(-1,-1)$, and with $y = x^2$ at $(1,1)$. $y = 1$ intersects with $y = x$ at $(1,1)$ and with $y = x^2$ at $(1,1)$ and $(-1,1)$. Finally, $y = x$ intersects with $y = x^2$ at $(1,1)$ and $(0,0)$. Careful counting reveals that there are $\boxed{12}$ pieces.

42. What is a_7? $a_7 = a_6 \times a_5 \times \cdots \times a_1 = 41$. $a_8 = a_7 \times a_6 \times a_5 \times \cdots \times a_1 = (a_7)^2 = \boxed{1681}$.

43. Young Guy needs to compute the sum of all products of integers ab, where $1 \leq a, b \geq 9$. This sum is equal to $(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)^2$. (Do you see why? Think about the distribution property!) The sum of all ab is therefore $45^2 = \boxed{2025}$.

44. $2^{2^4} - 1 = 2^{16} - 1 = (2^8 + 1)(2^8 - 1) = (257)(2^4 + 1)(2^4 - 1) = (257)(17)(2^2 + 1)(2^2 - 1) = (257)(17)(5)(3)$. This number has $\boxed{16}$ factors in total.

45. Let d_ℓ be the perpendicular distance from to line ℓ. $400 = AP^2 = d_{AB}^2 + d_{AD}^2$; $576 = BP^2 = d_{BA}^2 + d_{BC}^2$; $225 = CP^2 = d_{CB}^2 + d_{CD}^2$; we want $DP^2 = d_{DA}^2 + d_{DC}^2$. $400 + 225 - 476 = d_{AB}^2 + d_{AD}^2 + CP^2 = d_{CB}^2 + d_{CD}^2 - d_{BA}^2 - d_{BC}^2 = d_{DA}^2 + d_{DC}^2 = DP^2 = 49$. Thus, $DP = \boxed{7}$.

46. Note that $(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$, so $x^4 - 4x^3 + 2x^2 - 4x + 1 = (x-1)^4 - 4x^2 = ((x-1)^2)^2 - (2x)^2 = (x^2 - 4x + 1)(x^2 + 1) = 0$. So either $x^2 - 4x + 1 = 0 \implies x = 2 \pm \sqrt{3} \implies x - \frac{1}{x} = 2 \pm \sqrt{3} - (2 \mp \sqrt{3}) = \pm 2\sqrt{3} \implies y = 2\sqrt{3}$. $x^2 + 1 = 0 \implies x = \pm\sqrt{-1} \implies x - \frac{1}{x} = \pm\sqrt{-1} \pm \sqrt{-1} = \pm 2\sqrt{-1} \implies y = 2$.

47. Geometric sequences have the form, with first term a and ratio r, $a, ar, ar^2, \ldots$. The sum is $\frac{a}{1-r}$. So, we are given $\frac{a}{1-r} = 3$, $a + ar = \frac{8}{3}$. We see that $\frac{3}{8} = \frac{1}{a(1+r)}$, so multiplying the two equations gets $\frac{a}{1-r}\frac{1}{a(1+r)} = 3 \times \frac{3}{8} \implies \frac{1}{1-r^2} = \frac{9}{8} \implies 1 - r^2 = \frac{8}{9} \implies r = \pm\frac{1}{3}$. So now, plugging in, we have two possibilities: $a(\frac{4}{3}) = \frac{8}{3} \implies a = 2$, or $a(\frac{2}{3}) = \frac{8}{3} \implies a = 4$. $2 + 4 = \boxed{6}$.

48. We see that the number of intersections between the elements of A_i and A_j is $|A_i| \times |A_j|$. So, $34 = \Sigma(|A_i| \times |A_j|) = \frac{1}{2}((\Sigma|A_i|)^2 - \Sigma|A_i|^2)$. $\Sigma|A_i| = 10$, so $\Sigma|A_i|^2 = 32$. We now need to find a partition of 10 into parts whose square-sum is 32. After some careful work, we see that $(5, 2, 1, 1, 1)$ indeed satisfies this, so the answer is $5 \times 2 \times 1 \times 1 \times 1 = \boxed{10}$.

49. We use Vieta's formulas to see that $a + b = 52$ and $ab = 365$. Thus, $Q(x) = x^2 - 365x + 52$, so again by Vieta's $c + d = ab = 365$ and $cd = a + b = 52$. So, $a + b + c + d = 52 + 365 = \boxed{417}$.

50. It takes Rebecca $R(x,y) = \frac{|x|}{20} + \frac{|y|}{10}$ hours to reach point (x,y), and it takes The Great Bustard $GB(x,y) = \frac{\sqrt{x^2+y^2}}{10}$ hours to reach that same point. We wish to consider the set of points (x,y) with $GB(x,y) < R(x,y)$. We will only consider the first quadrant for the time being (as for $f = R$ or $f = GB$, $f(x,y) = f(-x,y) = f(x,-y) = f(-x,-y)$, so we can eliminate the $|\cdot$ signs).
We want $\frac{x}{20} + \frac{y}{10} > \frac{\sqrt{x^2+y^2}}{10} \implies \frac{x}{2} + y > \sqrt{x^2+y^2} \implies \frac{x^2}{4} + xy + y^2 < x^2 + y^2 \implies xy > \frac{3}{4}x^2 \implies y > \frac{3}{4}x$. We see that this region, in quadrant 1 and restricted to $x, y < 1$, has area $\frac{5}{8}$. A similar figure will occur in the other 4 quadrants, so the desired area is $\boxed{\frac{5}{2}}$.