Joe Holbrook Memorial Math Competition

5th Grade Solutions

October 15, 2017

1. Youjung scores a total of $2 \cdot 3 + 5 \cdot 4 + 2 \cdot 5 = 6 + 20 + 10 = \boxed{36}$ points.
2. It is 2 hours and 36 minutes from 3:00, and as there are 60 minutes in an hour, $2 \cdot 0 + 36 = \boxed{156}$.
3. Note there are 60 minutes in an hour and 60 seconds in a minute, so traveling 60 miles would take 60 minutes and $7 \cdot 60 = 420$ seconds, or $60 + 7 = \boxed{67}$ seconds.
4. If you take the number of spins and divide it by speed, you will get the time taken: $\frac{120 \text{ spins}}{6 \text{ spins per second}} = \boxed{20}$ seconds.
5. Since addition and subtraction come before division and mutiplication, $2 + 5 \cdot 2 - 5 = (2 + 5) \cdot (2 - 5)$. Now, since operations are performed right to left, $(2 + 5) \cdot (2 - 5) = 7 \cdot 3 = \boxed{21}$.
6. The larger square has side lengths of 6, which means that it has a perimeter of 24. Meanwhile, the smaller square, which has side lengths 3, has perimeter 12. Therefore, their collective perimeter is 36. Since the side of the smaller square is glued to the side of the larger square, you must subtract the edge between them, which is 3 for each square. Therefore, $36 - 6 = \boxed{30}$.
7. We factor 2670: $2670 = 2 \cdot 3 \cdot 5 \cdot 89$, so the largest prime factor is $\boxed{89}$.
8. $20\% \cdot 300 = 60$ of the donuts had sprinkles. Likewise, 30 had chocolate, and 111 had jelly. Thus, there were $300 - 60 - 30 - 111 = \boxed{99}$ plain donuts.
9. The battery will recharge at the rate of $2 - 0.5 = 1.5\%$ per minute. Because it needs to recharge 60%, the answer is $\frac{60}{1.5} = \boxed{40}$.
10. The distances form an arithmetic sequence, with a constant difference of 24. Using the formula $a_n = a_1 + (n-1)d$ and substituting in $a_1 = 120$, $n = 8$, and $d = 24$, we get $a_n = 120 + (8-1) \cdot 24$. Solving results in $a_n = \boxed{288}$.
11. Let x be the number of shots he must make to reach his goal. We have the inequality $\frac{7+x}{15+x} \geq 0.7$, or $x \geq \frac{35}{3}$. Since x must be a positive integer, the minimum valid value for x is $\boxed{12}$.
12. John does an odd number of pushups each minute. To reach a total of 100 pushups, he must do pushups for 10 minutes, as the sum of the first n odd numbers is n^2. Thus, he stops at $\boxed{\text{4:25 PM}}$.
13. Call the original number of boats x. The condition tells us that $6(x+1) = 9(x-1)$, which can be solved and $x = 5$. Thus the number of people is $6 \cdot (5+1) = \boxed{36}$.
14. We have to choose the remaining 2 members out of the 5 other people, so the answer is $\binom{5}{2} = \boxed{10}$.
15. Consider the straight segment between the two centers. 3 units of this is contained in one circle, and 5 units is contained in another. Therefore, the shortest distance from two points on each circle is $14 - 5 - 3 = \boxed{6}$.
16. Let x, y be the number of heads and tails, respectively. The condition fails if and only if $x < 500$ and $y < 500$. Summing the inequalities yields $x + y < 1000$, which is a contradiction. Thus, the probability that this occurs is 100%, so $n = \boxed{100}$.
17. Note that the units digit of 7^n repeats in cycles of 4. Since $\frac{2017}{4} = 504 + \frac{1}{4}$, the unit digit is the same as $7^1 = \boxed{7}$.

18. If I get an A+ (60%), then I have a 60% chance of getting a homework pass, which means that I have $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$ probability of getting a pass from here. If I don't get an A+ (40%), then I have a 20% chance, which means I have $\frac{2}{5} \cdot 15 = \frac{2}{25}$ chance of getting a homework pass from here. Adding them up yields a $\boxed{\frac{11}{25}}$ probability of getting the pass.

19. The given inequality can be factored as $(n-1)(n-2) \leq 0$. For this to hold, n must satisfy $1 \leq n \leq 2$. There are only two integers in this range, namely $\boxed{1, 2}$.

20. There are $\frac{7!}{2!}$ possible ways of rearranging Tiffany as there are 7 letters and the f is repeated twice. There are 6! combinations of rearranging Tifany as there are 6 letters and no repeated letters. $\frac{7!}{2!} = 2520$ and $6! = 720.2520 - 720 = \boxed{1800}$.

21. The numerator is $\frac{y-x}{4}$ and the denominator is $\frac{2(y-x)}{3}$, so the quotient is $\boxed{\frac{8}{3}}$.

22. Between any two circle, there can at most exist 2 intersections. Therefore, we have $2 \cdot \binom{17}{2}$. Between any two line, there can at most exist 1 intersection. Therefore, we have another $\binom{17}{2}$ intersections. Between a line and a circle, there can exist 1 intersection, for an additional we have $2 \cdot 17^2$intersections. This yields a total of $\boxed{986}$ intersections.

23. The condition is $2a^2 + b = b^2 + ab + b$, which simplifies to $(a-b)(2a+b) = 0$. Since a and b are different, $2a + b = 0$. $f(2)$ is $4 + 2a + b$, so the desired value is $\boxed{4}$.

24. Triangles $\triangle ABD$ and $\triangle BDC$ are equilateral since $\frac{120°}{2} = 60°$ so the total area is $2 \cdot \frac{6^2\sqrt{3}}{4} = \boxed{18\sqrt{3}}$

25. Looking at the hundredth place, the two digits have to be 9 and 1. Now, if the tenth place of the subtrahend is 1, the smallest it can be is 110, and the minuend would have to be more than 894+110=1004. Thus the tenth place of the subtrahend is $\boxed{0}$, and so is the product of the digits.

26. Imagine a square enclosing the octagon Side length of square would be $2 + \sqrt{2}$. To find the area, you can subtract 4 right isosceles triangle from the square: $(2+\sqrt{2})^2 - 1 \cdot 1/2 \cdot 4 = 4 + 2 \cdot \sqrt{2} + 2 - 2 = \boxed{4 + 2\sqrt{2}}$.

27. By properties of $30-60-90$ degree triangles, the height of the tree must be $6\sqrt{3}$ feet tall. As the triangle formed by the position of the bug, David, and the base of the tree forms another $30-60-90$ degree triangle, the height of the bug above the ground is $\frac{6}{\sqrt{3}} = \boxed{2\sqrt{3}}$ feet.

28. Listing out possibilities, we see that the 7 cases HHHTT, HHHHT, HHHTH, HHHHH, THHHT, THHHH, TTHHH, are the only possibilities, so the answer is $\boxed{\frac{7}{32}}$.

29. Chord BP is a diameter of Ω as point O lies on BP. As the radius of Ω is 2, the length of BP is 4, and the angle $\angle BCP = 90°$, applying the Pythagorean Theorem on $\triangle BCP$ yields $BC^2 + PC^2 = BP^2$. Plugging in the given values yields $BC^2 = 4^2 - 3^2 = 7 \to BC = \boxed{\sqrt{7}}$.

30. As $AB = 8$ and $BC = AD = 6$, by the Pythagorean Theorem, it follows that $BD = 10$. As M is the midpoint of BD, DM must be 5. As AN is the height to the base BD in triangle $\triangle ABD$, $\frac{1}{2} \cdot AN \cdot BD = \frac{1}{2} \cdot AB \cdot AD \to AN = \frac{24}{5}$. By the Pythagorean Theorem in triangle $\triangle AND$, $DN^2 = AD^2 - AN^2 \to DN = \frac{18}{5} \to MN = \boxed{\frac{7}{5}}$.

31. We look for patterns. The first multiple of 11 in the sequence is 55, which is the tenth term. The next two terms are 89 and 144, which both leave remainders of 1 when divided by 11. Since we are only concerned about when these numbers are divided by 11, these two numbers indicate that the pattern repeats, since the first two terms of the sequence are both 1's. Thus, for every 10 terms there is one multiple of 11, so the answer is $1000/10 = \boxed{100}$.

32. Each side of a hexagon intercepts a 60 degree arc of the circle, or $\frac{1}{6}$ of the circumference. Given a point on the circle, to choose another point on the circle such that the segment between does not intersect the hexagon requires that point to be part of the same arc. Therefore, there is a $\boxed{\frac{1}{6}}$ chance.

33. By the Legendre formula, the exponent of a prime p in the prime factorization of $g!$ is $\sum_{i=1}^{\infty} \lfloor \frac{g}{p^i} \rfloor$, so the exponent of 3 in the prime factorization of 12! is $\lfloor \frac{12}{3} \rfloor + \lfloor \frac{12}{9} \rfloor + \ldots = 4+1+0+\ldots = 5$, while the exponent of 5 is $\lfloor \frac{12}{5} \rfloor + \lfloor \frac{12}{25} \rfloor + \ldots = 2 + 0 + \ldots = 2$. Therefore $k \leq 5$ and $k \leq 2$, so the maximum is reached at $k = \boxed{2}$.
Alternatively, consider that the distribution of 3 is more common than 5, so we only need to evaluate the exponent of 5.

34. Since 100 is divisible by 5, every fifth person dies and we are left with every number that is $2, 3, 4$, or 0 mod 5. The next round starts at 2 mod 5, and will proceed to kill a 3 mod 5, and so on. This leads to getting $\boxed{87}$ as a final answer.

35. Note that for any root r_i, then $r_i^{100} = 12r_i - 3$. Hence $r_1^{100} + r_2^{100} + \ldots + r_{100}^{100} = 12(r_1 + r_2 + \ldots + r_{100}) - 300$. But by Vieta's formulas, the sum of the roots is 0, hence the answer is $\boxed{-300}$.

36. By power of a point, we know $PB \cdot PC = PA^2$.Since AC is a diameter, and PA is a tangent, we know that $AC \perp PA$. Thus we can use the Pythagorean theorem to solve for PC.
We have $PA^2 + AC^2 = PC^2$. Substituting $PB \cdot PC = 16 \cdot PC$ for PA^2, and rearranging, we get $PC^2 - 16PC - 15^2 = 0$. Solving for PC and taking the positive root we get $\boxed{25}$.

37. Pick $x_1, x_2, \ldots, x_{99}$ arbitrarily, for a total of $\boxed{100^{99}}$ possibilities. Let the remainder of their sum is divided by 100 be S. Then x_{100} must be equal to $100 - S$, which is uniquely determined for each choice of $x_1, x_2, \cdots, x_{99}$.

38. Squaring both sides, we get $x^2 + y^2 + z^2 + 2xy + 2yz + 2xz = 2xy + 2yz + 2xz + 41$. The $2xy + 2yz + 2xz$ terms cancel, and the resulting equation is $x^2 + y^2 + z^2 = 41$. There are 9 solutions where $x, y, z \neq 0$: 6 permutations of $1 + 4 + 36$ and 3 permutations of $9 + 16 + 16$, and 6 solutions with exactly one of x, y, z equal to 0: $16 + 25$. Therefore there are $\boxed{15}$ total solutions.

39. Let the center of the circle be O. WLOG, we consider $\triangle OPQ$. Let the foot of the perpendicular from O to BC be M, so $BM = \frac{1}{2}BQ = 3$. Since $OP = 5$, from right triangle OBM, we have $OM = 4$. Identically, if the feet of the perpendiculars from O to BC and CA are L and N, then $OL = ON = 4$. Thus, O is the incenter of $\triangle ABC$.

    Let $AB = x$. Recall that the area of $\triangle ABC$ is rs, where r is the inradius and s is the semiperimeter, and is also $\frac{24x}{2} = 12x$. Thus,

$$[\triangle ABC] = rs = 4\frac{\left(24 + x + \sqrt{24^2 + x^2}\right)}{2} = 2\left(24 + x + \sqrt{576 + x^2}\right),$$

$$[\triangle ABC] = 12x,$$

$$2\left(24 + x + \sqrt{576 + x^2}\right) = 12x,$$

$$24 + x + \sqrt{576 + x^2} = 6x,$$

$$\sqrt{576 + x^2} = 5x - 24$$

$$576 + x^2 = 25x^2 - 240x + 576,$$

$$240x = 24x^2,$$

$$240 = 24x,$$

$$x = 10.$$

    Hence, $AB = \boxed{10}$.

40. Let E be the expected value of P. The expected value of any roll of the die is $\frac{1+2+3+4+5+6}{6} = \frac{7}{2}$, so $a = b = c = d = \frac{7}{2}$. It follows that $E(1) = 7, E(2) = \frac{49}{4}, E(3) = 7, E(4) = \frac{49}{4}$.

Since P has degree at most 3, E does as well, so its third finite differences are constant. Note that its first finite differences are

$$E(2) - E(1), E(3) - E(2), E(4) - E(3), E(5) - E(4), ...,$$

its second finite differences are

$$E(3) - 2E(2) + E(1), E(4) - 2E(3) + E(2), E(5) - 2E(4) + E(3), ...,$$

so its third finite differences are

$$E(4) - 3E(3) + 3E(2) - E(1), E(5) - 3E(4) + 3E(3) - E(2), ...$$

Thus,

$$\begin{aligned} E(4) - 3E(3) + 3E(2) - E(1) &= E(5) - 3E(4) + 3E(3) - E(2), \\ E(5) &= 4E(4) - 6E(3) + 4E(2) - E(1) \\ E(5) &= 4\left(\frac{49}{4}\right) - 6(7) + 4\left(\frac{49}{4}\right) - 7 \\ &= 49 - 42 + 49 - 7 = \boxed{49}. \end{aligned}$$