Joe Holbrook Memorial Math Competition

6th Grade Solutions

October 14, 2018

1. $80 \text{ problems} \cdot \dfrac{1 \text{ minute}}{5 \text{ problems}} = \boxed{16}$ minutes.

2. We use PEMDAS: $2 \div (0+1) \cdot 8 = 2 \div 1 \cdot 8 = 2 \cdot 8 = \boxed{16}$.

3. Jenn ate $\dfrac{2}{3} \cdot \dfrac{2}{5} = \dfrac{4}{15}$ of the cake. Therefore, Shalin and Jenn together ate $\dfrac{4}{15} + \dfrac{1}{3} = \dfrac{9}{15}$ of the cake. Thus, $1 - \dfrac{9}{15} = \dfrac{6}{15} = \boxed{\dfrac{2}{5}}$ of the cake remained.

4. Working backwards, we multiply 130 by 4 to find the total number of students: $130 \times 4 = \boxed{520}$.

5. Multiply to find the answer of $4 \cdot 10 \cdot 18 = \boxed{720}$ dollars.

6. On day n, the student solves $3n+1$ problems, since on day 1 she solved $3 \times 1 + 1 = 4$ problems and she solves three more problems than she did the day before. On day 12, she solves $3 \times 12 + 1 = \boxed{37}$ problems.

7. There are three primes between 1 and 6, inclusive: 2, 3, and 5. Therefore, the answer is $\dfrac{3}{6} = \boxed{\dfrac{1}{2}}$.

8. Let x be Mikako's favorite positive number. Then,

$$\sqrt{4x-11} = 3 \implies (\sqrt{4x-11})^2 = 3^2 \implies 4x - 11 = 9 \implies x = 5.$$

   Thus, Mikako's favorite positive number is $\boxed{5}$.

9. Each chopped off head results in 2 more heads. Therefore, the answer is $2 \cdot 7 + 20 = \boxed{34}$.

10. The factors of 2018 are 1, 2, 1009, and 2018. Thus, the sum of the divisors of 2018 is $1+2+1009+2018 = 3030$. The prime factorization of 2019 is $3^1 \cdot 673^1$. Thus, 2019 has 4 factors. The answer is $3030 - 4 = \boxed{3026}$.

11. Since 2019 is odd, it cannot be the sum of two odd prime numbers. Therefore, $2019 = 2 + 2017$ is the only possibility, and there is exactly $\boxed{1}$ way.

12. From the formula, we have

$$2 \star 1 = 5(2) + 3(1) + 1 \star 1 = 23 \implies 2 \star 2 = 5(2) + 3(2) + 2 \star 1 = \boxed{39}.$$

13. Initially, 180 Starbursts are pink and 420 are not pink. When one removes pink Starbursts only, the number of Starbursts which are not pink remains fixed at 420. If the number of not-pink Starbursts is to account for 80% of the bag after the removal, then there must be a total of $\dfrac{420}{0.8} = 525$ Starbursts at the end of the procedure. Hence, $\boxed{75}$ pink Starbursts must be removed.

    (Alternatively, one can also use the equation $\dfrac{180-x}{600-x} = 0.2$ to represent the fact that once x pink Starbursts have been removed, 20% of the remaining Starbursts are pink.)

14. $4^{1010} \cdot 5^{2018} = 2^{2020} \cdot 5^{2018} = 4 \cdot 10^{2018}$, which has $\boxed{2019}$ digits.

15. Let d denote the number of dogs and b denote the number of birds. Since each animal has one head, the total number of heads is the same as the number of birds and dogs. Thus, $d + b = 76$.

    Since dogs have 4 legs and birds have 2, $4d+2b = 232$. Therefore, we have a system of two linear equations in two variables:
    $$\begin{cases} d + b = 76 \\ 4d + 2b = 232 \end{cases}$$
    Dividing the second equation by 2, we find $2d+b = 116$. We subtract the first equation from this equation to find $d = 2d + b - (d + b) = 116 - 76 = 40$. Hence, there are a total of $\boxed{40}$ dogs.

16. For $n \geq 2$, the probability that domino n is knocked over is $\frac{n}{n+1}$. Furthermore, for $n \geq 2$, domino n cannot be knocked over unless domino $n-1$ is knocked over. Therefore, the answer is $\frac{2}{3} \cdot \frac{3}{4} \cdots \frac{2018}{2019} = \boxed{\frac{2}{2019}}$.

17. From the given information, $\frac{\frac{n}{2}+7}{n+7} = \frac{3}{5}$. Cross-multiplying, we find $\frac{5}{2}n + 35 = 3n + 21$. Solving this equation yields the answer of $n = \boxed{28}$.

18. The sum of Andrew's scores on the five tests he has taken is $94 \cdot 5 = 470$. In order to receive an average of 95 across all six tests, the sum of his scores on the six tests must be at least $6 \cdot 95 = 570$. Therefore, he must receive a score of (at least) $570 - 470 = \boxed{100}$ on his sixth test.

19. The 48 couples contain a total of $48 \times 2 = 96$ people. The probability a randomly selected person is not in a couple is therefore $\frac{120-96}{120} = \frac{24}{120} = \boxed{\frac{1}{5}}$.

20. There are $1 + 2 + 3 + ... + 10 = \frac{10 \cdot 11}{2} = 55$ cards in the deck. As there are six 6's, the probability that Autumn draws a 6 is $\boxed{\frac{6}{55}}$.

21. Since the ratio of side-lengths of the two pentagons is $\frac{3}{5}$, the ratio of the areas of the two pentagons is $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$. Let the area of the smaller pentagon be x. Then, $\frac{9}{25} = \frac{x}{35} \implies x = 35 \cdot \frac{9}{25} = \boxed{\frac{63}{5}}$.

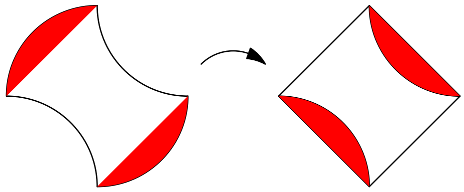
22. The first sentence of the problem is irrelevant! Since Lyp randomly calls each throw, he has a 50% chance of calling each throw correctly. As there are $8 = 2^3$ total outcomes with equal probability, and as only 3 of them have exactly 2 correct calls, there is a $\boxed{\frac{3}{8}}$ chance that Lyp calls exactly 2 out of the 3 free throws correctly.

23. There are nine digits to choose from. (There cannot be a zero in the number because, by the condition, zero would have to be the first digit, and a number cannot begin with zero.) For each subset of six digits from 1 through 9, there is exactly one way to arrange them in increasing order. Thus, there are $\binom{9}{6} = \boxed{84}$ 6-digit numbers with their digits in increasing order.

24. First, note that the number of people in the room must be divisible by 40. Let the number of the people in the room equal to $40k$, for some positive integer k. Then exactly $25k$ people like chocolate and exactly $24k$ people like vanilla. This means at least $24k + 25k - 40k = 9k$ people like both (if any fewer liked both, then the total number of people would exceed $40k$). Thus, the smallest number of people who like both chocolate is $\boxed{9}$, achieved when $k = 1$. In this case, 9 people like both chocolate and vanilla, 16 only like chocolate, and 15 only like vanilla.

25. Let Ben's building rate, Eddie's building rate, and Harahm's building rate be B, E, and H, respectively (in fractions of the birdhouse per day). By the relationship rate $= \frac{\text{progress}}{\text{time}}$, $B + E = \frac{1}{3}$, $B + H = \frac{1}{4}$, and $H + E = \frac{1}{5}$. Adding these three equations and dividing by two yields $B + H + E = \frac{47}{120}$. This means that the rate of them working together to build the birdhouse is $\frac{47}{120}$. Thus, it would take them $\boxed{\frac{120}{47}}$ days to complete the birdhouse when working together.

26. The region that the lamb can wander in is a circle of radius 6 around the origin, with two circular cutouts of radius 6 from each of the two wolves. It can be noted that if you move the cut-out pieces properly, you can form a square with a diagonal of length 12. Therefore, the area of the region is $\boxed{72}$.



27. The probability that the compulsive liar tells more truths than lies is the same as the the probability that he tells more lies than truths. Note that there cannot be an equal number of lies and truths, since there is an odd number of questions in total. Therefore, the desired probability is $\boxed{\frac{1}{2}}$.

28. The common difference is irrelevant! It's well known that the sum of the terms of an arithmetic sequence is the average of the first and last terms multiplied by the number of terms. In this case, the average of the first and last terms (their sum divided by 2) is 25. Therefore, the answer is $25 \cdot 3000 = \boxed{75000}$.

29. Each player plays 3 games within his or her group. Thus, $\binom{4}{2} = 6$ games occur in each group. Furthermore, each group eliminates 3 people and sends 1 person to the next round. Thus, since the tournament lasts until a single person survives (i.e until 1023 of them are eliminated), there will be $\frac{1023}{3} = 341$ groups formed throughout the game (since each group eliminates exactly 3 players). Thus, a total of $6 \cdot 341 = \boxed{2046}$ games will take place throughout the tournament.

30. The sum $\frac{p}{q} + \frac{q}{p}$ can be rewritten as $\frac{p^2+q^2}{pq} = \frac{(p+q)^2 - 2pq}{pq}$. By Vieta's Formulas, $p + q = -5$ and $pq = -17$. Plugging these values into the above fraction yields the answer of

$$\frac{(p+q)^2 - 2pq}{pq} = \frac{25+34}{-17} = \boxed{-\frac{59}{17}}.$$

31. Let t be the amount of time it takes (in hours) for the trains to meet. Then $360 = 60t + 60t \implies t = \frac{360 \text{ miles}}{120 \text{ miles/hour}} = 3$ hours. Therefore, the crow flies for 3 hours before the trains meet. Hence, the crow travels a total distance of 40 miles/hour $\times$ 3 hours $= \boxed{120 \text{ miles}}$ before the trains meet.

32. The value of ϕ is irrelevant! The given expression can be rewritten as $\frac{(x+\phi)^2 + 2017}{2018}$, which has a minimum value of $\boxed{\frac{2017}{2018}}$ at $x = -\phi$.

33. Call moving up one unit and left one unit a *type I move,* and call moving up one unit and right one unit a *type II move.* Then, Elaine must make 2 type I moves and 3 type II moves. She can make these moves in any order. Out of her 5 total moves, 2 of them must be of type I, so there are a total of $\binom{5}{2} = \boxed{10}$ ways she can move to $(1, 5)$.

34. In order to solve for the coordinates of points A and B, we need to solve the equation $x^2 + 6x + 8 = -x^2 - x + 2$. Moving the terms to the left-hand side yields $2x^2 + 7x + 6 = 0$. The quadratic formula yields roots $x = -2$ and $x = -\frac{3}{2}$. Plugging these x-values back into $y = x^2 + 6x + 8$, we find that the coordinates of A and B are $(-2, 0)$ and $\left(-\frac{3}{2}, \frac{5}{4}\right)$. Applying the formula for slope gives an answer of $\frac{\frac{5}{4} - 0}{-\frac{3}{2} - (-2)} = \boxed{\frac{5}{2}}$.
Note: The order in which you calculate the slope of a line does not affect your answer. You would find the same answer of $\frac{5}{2}$ if you calculated the slope as $\frac{0 - \frac{5}{4}}{-2 - (-\frac{3}{2})}$, since the minus signs cancel out.

35. Since we only care about individual digits, we may treat each number as a string of digits. Since we wish to count the total number of 9's, we can add leading 0's so that every number in the list has exactly 4 digits. (For example, 0 becomes 0000, 1 becomes 0001, and so on.) Thus, we want to count the number of 9's in the list

$$0000,\ 0001,\ 0002,\ \dots,\ 9999.$$

Now, note that all the strings of the form $\overline{abcd}$, with $0 \le a, b, c, d \le 9$, are present in the list. Therefore, every digit appears the same number of times on the list. Since there are 10000 numbers in the list, there are a total of 40000 digits. Hence, the digit 9 appears $\frac{40000}{10} = \boxed{4000}$ times.

36. We can rewrite all of the roots as pure exponents. The infinite product becomes

$$2^{\frac{1}{1\cdot 2}} \cdot 2^{\frac{1}{2\cdot 3}} \cdot 2^{\frac{1}{3\cdot 4}} \cdots.$$

We can combine all these powers of 2 to find

$$2^{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots}.$$

We now notice that the exponent telescopes to 1 (as $\frac{1}{1\cdot 2} = \frac{1}{1} - \frac{1}{2}$, $\frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$, and so on). Therefore, the infinite product equals $2^1 = \boxed{2}$.

37. Let point E denote the intersection of the diagonals AC and BD. Furthermore, let points F and G denote the midpoints of AB and CD, respectively. It can be shown that the midpoint of the hypotenuse of a right triangle is equidistant from each of the vertices of the triangle. Thus, $FE = \frac{63}{2}$ and $GE = \frac{149}{2}$. Now, note that $\triangle FBE$ is isosceles with $FB = FE$. Similarly, $\triangle EGD$ is isosceles with $GD = GE$. Therefore, $\angle DEG = \angle DGE = \angle FBE = \angle FEB$ (in fact, all of them are 37.5°). Thus, $\angle DEG = \angle FEB$ and D, E, B are collinear $\implies F, E, G$ are collinear. Therefore, $FG = FE + EG = \boxed{106}$.

38. Label each square with the number of ways in which it is possible to reach that square. The top row contains one 1, and for rows 2 through 4, each of the numbers in the row is equal to the product of the number in the previous row and the number of squares in the previous row. (This is because one can choose to move down on any cell in the row.) For rows 5 and 6, each of the numbers in the row equals the product of the number in the previous row and the number of squares in the row itself (since one can only move down onto one of the squares in the row). Therefore, there are $1 \times 2 \times 4 \times 6 \times 4 \times 2 = \boxed{384}$ ways to get to the bottom right square.

		1	1		
	2	2	2	2	
8	8	8	8	8	8
48	48	48	48	48	48
	192	192	192	192	
		384	384		

39. We recognize the powers of three in the problem, and rewrite the given polynomial as $x^9 + 3^5x^3 + 3^6$. As the first and last terms are both cubes, we try to "complete the cube". Lo and behold, $(x^3 + 9)^3 = x^9 + 27x^6 + 243x^3 + 729$. Therefore, our polynomial is simply $(x^3 + 9)^3 - 27x^6$. This expression is a difference of cubes, which factors as $(x^3 - 3x^2 + 9)q(x)$, for some polynomial $q(x)$ of degree 6. Hence, the answer is $\boxed{x^3 - 3x^2 + 9}$.

40. Suppose that the number t has period 2. Therefore, it satisfies $P(P(t)) = t$, or, equivalently, $P(P(t)) - t = 0$. Moreover, it does not satisfy $P(t) - t = 0$, otherwise, it would have period 1. Thus, t satisfies $\frac{P(P(t))-t}{P(t)-t} = 0$. Substituting the expression for P and expanding yields $t^2 + 3t + 3$, whence by Vieta's Formulas, it follows that the answer is $\boxed{-3}$.