# Joe Holbrook Memorial Math Competition

5th Grade Solutions

October 13, 2019

1. Using PEMDAS, $2 \times (0 - 1) + 9 = 2 \times (-1) + 9 = -2 + 9 = \boxed{7}$.

2. Recall that the formula for the area of a rectangle is length times width. As the area is 20 and the width is 5, the length is $20/5 = \boxed{4}$.

3. There are 7 days in a week. Thus the number of pounds of potatoes generated per day is $119/7 = \boxed{17}$.

4. We calculate Simon's height to be 67 inches and Andrew's height to be 76 inches. Subtracting, we find $76 - 67 = \boxed{9}$.

5. Simply perform division. We find $\boxed{5.75}$.

6. After simplification the equation becomes $12x = 24$. Solving for $x$, we find $x = \boxed{2}$.

7. Note that each corresponding pair of digits sum to 10. As there are 8 digits after the decimal point, the answer is $\boxed{11.1111111}$ (with 7 ones after the decimal point).

8. Since 2017, 2018, and 2019 form an arithmetic progression, $2017 + 2018 + 2019 = 3 \cdot 2018$. Hence the answer is $\dfrac{3 \cdot 2018}{6} = \dfrac{2018}{2} = \boxed{1009}$.

9. Note that 5.1 hours equals $5.1 \times 60 = 306$ minutes. Since Charles recited 100 words per minute, in the 5.1 hours, Charles recited $306 \times 100 = \boxed{30600}$ words, which is our answer.

10. As $2 \cdot 5 = 10$ divides $9!$, the last digit is $\boxed{0}$.

11. It is given that 3 and 5 are both less than 4. Further, $4 < 1 < 2$. Thus, $\boxed{4}$ is the median.

12. Let $x$ be the answer to this question. Then $x = 1009 + \dfrac{x}{2019}$, implying $x = \boxed{1009.5}$.

13. Let the three integers be $n - 1, n, n + 1$. Then their sum is $(n - 1) + n + (n + 1) = 3n = 15$, and hence the middle integer is $\frac{15}{3} = 5$. Since the three integers are 4, 5, and 6, their product is $4 \cdot 5 \cdot 6 = \boxed{120}$.

14. When a number $x$ is entered into the machine, the machine adds 6, which gives $x + 6$. Next, it multiplies by 2 and subtracts 4, which yields $2(x+6) - 4$. We are given that $2(x+6) - 4 = 14$. Solving this equation yields $x = \boxed{3}$.

15. Note that $\langle 6 \rangle = 1+2+3 = 6$, hence $\langle\langle 6 \rangle\rangle = \langle 6 \rangle = 6$, and $\langle\langle\langle 6 \rangle\rangle\rangle = \langle 6 \rangle = \boxed{6}$. (Fun fact: in number theory, we call $n$ a *perfect number* if $\langle n \rangle = n$.)

16. Using conversion factors, we find the answer of $100 \text{ flips} \cdot \dfrac{3 \text{ flaps}}{2 \text{ flips}} \cdot \dfrac{6 \text{ flops}}{5 \text{ flaps}} = \boxed{180}$ flops.

17. We use PEMDAS. We have $3 \circ 2 = 3^2 - 2^2 = 9 - 4 = 5$. This leaves us with $7 \circ 5$, or $7^2 - 5^2 = 49 - 25 = \boxed{24}$.

18. Jerry quacks at times that are multiples of 10 minutes after 12:00, and Akash quacks at times that are multiples of 18 minutes after 12:00. They will quack at the same time at multiples of both 10 and 18 minutes after 12:00. The least common multiple of 10 and 18 is 90. Thus, they will both quack 90 minutes after 12:00, at $\boxed{\text{1:30 PM}}$.

19. To calculate the exponent tower, we first calculate $1^9 = 1$, then $0^{1^9} = 0^1 = 0$, and finally $2^{0^{1^9}} = 2^0 = 1$. Hence, the original expression equals $1 \times 2018 - 2017 = 2018 - 2017 = \boxed{1}$, our answer.

20. Let $x$ be the number of people who voted twice. Let $y$ be the number of people who voted once. We get the system of equations $x + y = 2020$ and $2x + y = 2019 + 2018$. Solving this yields $x = \boxed{2017}$.

21. Since $x$ must be real, $k \geq 0$. Since $x$ must be an integer, $k$ must be a square. There are $\boxed{10}$ squares from 0 to 99 inclusive.

22. Isolating the sum $a^4 + b^4$, we find $a^4 + b^4 = 97$. Clearly, $a$ and $b$ must be less than 4. Testing $a = 3$, we find $b^4 = 97 - 3^4 = 16 \implies b = 2$. By symmetry the pair $(a, b) = (2, 3)$ satisfies the equation. Either way, the answer is $a + b = \boxed{5}$.

23. Given $2^x = 25$, we raise both sides of the equation to the $\frac{1}{2}$ power, yielding $2^{\frac{x}{2}} = 5$. If we want $2^{\frac{x}{2}+3}$, then we should multiply both sides of the equation by $2^3 = 8$. Thus, $2^{\frac{x}{2}+3} = 5 \cdot 8 = \boxed{40}$.

24. *Method 1:* The sum of the coefficients of a polynomial $P(x)$ is simply $P(1)$. To see this, note that if $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then

$$P(1) = a_n 1^n + a_{n-1}1^{n-1} + \ldots + a_1 1 + a_0$$

$$= a_n + a_{n-1} + \ldots + a_1 + a_0,$$

precisely the sum of the coefficients of $P(x)$. Hence, the answer is $(2 \cdot 1 + 1)^3 = 3^3 = \boxed{27}$.

*Method 2:* Expanding the given polynomial, we find

$$(2x + 1)^3 = (2x + 1)^2(2x + 1) = (4x^2 + 4x + 1)(2x + 1)$$

$$= 8x^3 + 12x^2 + 6x + 1.$$

Thus the sum of the coefficients is $8 + 12 + 6 + 1 = 27$, as above.

25. The product is divisible by $2 \cdot 5 = 10$, so the remainder is either 0 or 10. The remainder can't be 0, as the product is only divisible by one power of 2. Hence our answer is $\boxed{10}$.

26. Let $c$ denote the number of chickens on the farm, and let $p$ denote the number of pigs on the farm. Since there are 21 heads total, including Eric's, $c + p + 1 = 21$. Also, since there are 52 feet total, including both of Eric's, $2c + 4p + 2 = 52$. Solving this system, we find the answer of $c = \boxed{15}$ chickens.

27. The first two sentences are enough to solve the problem. The difference between two prime numbers can only be odd if the smaller number is 2. Thus Eric was $\boxed{2}$ years old and Charles was 17.

28. Note that $a$ can be negative. Hence, we take $a$ to be the most negative it can be, so long $b$ remains even. The least possible value of $b$ is 2, which corresponds to the most negative value of $a = -128$, yielding a sum of $-128 + 2 = \boxed{-126}$.

29. Recall that if a number is divisible by 9, the sum of digits of the number must be divisible by 9. As 9 divides $x$, 9 must divide $2+5+9+1+A+0+B = 17+A+B$, implying that $A+B$ leaves a remainder of 1 when divided by 9. Similarly, as 9 divides $y$, 9 must divide $1+0+2+4+2+A+B+C = 9+A+B+C$, implying that $A + B + C$ leaves a remainder of 0 when divided by 9. It follows that $C$ must leave a remainder of 8 when divided by 9. As $C$ is a digit between 0 and 9, $C$ must equal $\boxed{8}$, our answer.

30. Note that $|x - y| = \sqrt{(x - y)^2}$. Hence, to find $|x - y|$, it suffices to compute $(x - y)^2$. To finish, note that

$$\begin{aligned}(x - y)^2 &= x^2 - 2xy + y^2 \\ &= x^2 + 2xy + y^2 - 4xy \\ &= (x + y)^2 - 4xy \\ &= 37 - 3 \cdot 4 \\ &= 25,\end{aligned}$$

thus $|x - y| = \boxed{5}$.

31. Let $S, A$, and $H$ denote the number of students in the Science, Arts, and Humanities departments, respectively. Note that 4 must divide $S$, 3 must divide $A$, and 2 must divide $H$. Thus, the least possible values for $S, A$, and $H$ are 4, 3, and 2, respectively. The given condition states that $\frac{S}{4} + \frac{A}{3} + \frac{H}{2} = \frac{S + A + H}{3}$. By inspection, the triple $(S, A, H) = (4, 3, 2)$ satisfies this condition, and hence the smallest possible number of students at Barry's College of Academics is $4 + 3 + 2 = \boxed{9}$.

32. Since there are four seats around the table, Simon must sit across from Doug. Since Doug must sit next to Sumner, Sumner must either sit to the right or left of Doug. If Sumner sits to the right of Doug, then Andy must sit to the left of Doug, and similarly, if Sumner sits to the left of Doug, then Andy must sit to the right of Doug. These two cases are rotationally distinct, and hence there are $\boxed{2}$ ways to seat the four people.

33. Note that $V_1 = (0, 0)$ and $V_2 = (0, 4)$. To find the intersection points $A, B$, we solve $x^2 = 4 - x^2 \implies x^2 = 2 \implies x = \pm\sqrt{2}$. Then the intersections are $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$. Thus, $A, B$ are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$ in some order. Since these points form a rhombus by symmetry, we can simply find the length of one of the sides and multiply by 4. As the distance between $(\sqrt{2}, 2)$ and $(0, 0)$ is $\sqrt{2+4} = \sqrt{6}$, the answer is $\boxed{4\sqrt{6}}$.

34. Let the isosceles triangle have sides $a$, $a$ and $b$. As the perimeter is 60, $2a + b = 60$. Further, the triangle inequality implies $2a > b$. Hence, $60 = 2a + b > 2b \implies b < 30$. As $b = 60 - 2a$ must be even, $b$ can be any even integer between 2 and 28, inclusive. Each of these 14 values of $b$ yields a non-degenerate isosceles triangle, and hence, our answer is $\boxed{14}$.

35. We have $A = 3 + \frac{1}{A}$, hence $A^2 - 3A - 1 = 0$. Solving gives us $A = \frac{3 \pm \sqrt{13}}{2}$. Since $\sqrt{13} > 3$, we must choose the positive solution, implying $A = \frac{3+\sqrt{13}}{2}$. Similarly, $B = \sqrt{3+B}$, hence $B^2 - B - 3 = 0$ and $B = \frac{1+\sqrt{13}}{2}$. Thus, $A - B = \boxed{1}$.

36. Notice that each of the coefficients of the polynomial, as well as the right hand side, are divisible by 10001. Dividing the equation by 10001, we are left with $2x^3 + x + 9 = 2019$. We recognize that using base 10 expansion, $2019 = 2 \cdot 10^3 + 1 \cdot 10 + 9$, and hence our answer is $x = \boxed{10}$.

37. By the Pythagorean Theorem on $\triangle ABC$, $AC = \sqrt{3^2 + 4^2} = 5$. By the Pythagorean Theorem on $\triangle CDA$, $DA = \sqrt{5^2 - 1^2} = 2\sqrt{6}$. Since $\triangle ABC$ and $\triangle CDA$ are right triangles, their areas are $\frac{AB \cdot BC}{2} = 6$ and $\frac{CD \cdot DA}{2} = \sqrt{6}$, respectively. Since the area of $ABCD$ equals the sum of the areas of $\triangle ABC$ and $\triangle CDA$, our answer is $\boxed{6+\sqrt{6}}$.

38. Notice that if the sum of the digits of a number is 9, then it must be divisible by 9 by the divisibility rule for 9. Therefore, Susanian integers have the property that they are divisible by 9 and 15. This is equivalent to being divisible by their least common multiple which is 45. However, it is important to realize that not all numbers that are divisible by 9 have sum of digits 9 - it could have a sum of digits that is a multiple of 9. Therefore, we test all multiples of 45 less than 500 to see if they have sum of digit 9. These multiples are $45, 90, 135, 180, 225, 270, 315, 360, 405, 450, 495$. Notice that $\boxed{10}$ of these are Susanian but 495 has sum of digit 18 and is not.

39. Note that for all $k \geq 0$, $2^{2k}$ leaves a remainder of 1 when divided by 3, while $2^{2k+1}$ leaves a remainder of 2 when divided by 3. Thus, $\left\lfloor \frac{2^{2k}}{3} \right\rfloor = \frac{2^{2k}-1}{3}$ and $\left\lfloor \frac{2^{2k+1}}{3} \right\rfloor = \frac{2^{2k+1}-2}{3}$, implying that $\left\lfloor \frac{2^{2k}}{3} \right\rfloor + \left\lfloor \frac{2^{2k+1}}{3} \right\rfloor = \frac{2^{2k}+2^{2k+1}}{3} - 1$ for all $k \geq 0$. Since the desired sum equals the sum of $\left\lfloor \frac{2^{2k}}{3} \right\rfloor + \left\lfloor \frac{2^{2k+1}}{3} \right\rfloor$ for $0 \leq k \leq 5$, it follows that the desired sum equals

$$\left(\frac{2^0+2^1}{3} - 1\right) + \left(\frac{2^2+2^3}{3} - 1\right) + \ldots + \left(\frac{2^{10}+2^{11}}{3} - 1\right)$$

$$= \frac{2^0 + 2^1 + \ldots + 2^{11}}{3} - 6 = \frac{2^{12}-1}{3} - 6 = \boxed{1359}.$$

40. The area of the parallelogram with vertices $(0, 0), (x, 1), (1, x)$, and $(x + 1, x + 1)$ is $x^2 - 1$. Thus,

$$\frac{1}{f(x)} = \frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right).$$

Thus, the desired infinite sum equals

$$\frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots\right).$$

This sum telescopes to $\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right)=\frac{1}{2}\cdot\frac{3}{2}=\boxed{\frac{3}{4}}$, which is the answer.